Towards explicit realizations of the Sato-Tate groups of abelian threefolds

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Zeta functions of algebraic varieties

For X an algebraic variety over a finite field $\mathbb{F}_q$, the **zeta function** of X is

$$Z(X,T) = \prod_{x \in X^\circ} (1 - T^{\deg(x/\mathbb{F}_q)})^{-1} = \exp\left(\sum_{n=1}^\infty \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right),$$

where X° denotes the closed points of X (i.e., Galois orbits of $\overline{\mathbb{F}}_q$-points).

For X smooth proper over $\mathbb{F}_q$, we have

$$Z(X,T) = \frac{P_1(T) \cdots P_{2g-1}(T)}{P_0(T) \cdots P_{2g}(T)}$$

where $P_i(T)$ is (the reverse of) a q^i-**Weil polynomial**:

- $P_i(T)$ has integer coefficients and its constant term is 1.
- The roots of $P_i(T)$ in $\mathbb{C}$ all lie on the circle $|T| = q^{-i/2}$.

Curves and abelian varieties

When X is a (smooth, proper, geometrically integral) curve of genus g,

$$P_0(T) = 1 - T, \qquad P_2(T) = 1 - qT,$$

$P_1(T)$ is of degree $2g$, and $P_1(q^{-1/2}T)$ is palindromic.

When X is an abelian variety of dimension g, $P_1(T)$ is of degree $2g$, $P_1(q^{-1/2}T)$ is palindromic, and $P_i(T) = \wedge^i P_1(T)$. That is, if P_1 has roots $\alpha_1, \dots, \alpha_{2g}$, then P_i has roots

$$\alpha_{j_1} \cdots \alpha_{j_i} \qquad (1 \leq j_1 < \cdots < j_i \leq 2g).$$

The values of P_1 for a curve and its Jacobian coincide.

L-functions

For A an abelian variety over a number field K with ring of integers $\mathfrak{o}_K$, its **(incomplete)** ***L*-function** is the Dirichlet series

$$L(A,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\mathrm{Norm}(\mathfrak{p})^{-s})^{-1}$$

where $\mathfrak{p}$ runs over prime ideals of $\mathfrak{o}_K$ at which A has good reduction, $\mathrm{Norm}(\mathfrak{p}) = \#(\mathfrak{o}_K/\mathfrak{p})$ is the absolute norm, and $L_{\mathfrak{p}}(T)$ is the factor $P_1(T)$ of the zeta function of the reduction of A modulo $\mathfrak{p}$.

For example, if A is an elliptic curve over $\mathbb{Q}$, this is the usual expression

$$L(A,s) = \prod_p (1 - a_p p^{-s} + p^{1-2s})^{-1}, \qquad a_p = p + 1 - \#A(\mathbb{F}_p).$$

In general, $L(A,s)$ converges absolutely for $\mathrm{Re}(s) > 3/2$ but is expected to admit a meromorphic continuation to $\mathbb{C}$.

Distribution of Euler factors

With the functional equation in mind, we renormalize the L-polynomials:

$$\overline{L}_{\mathfrak{p}}(T) = L_{\mathfrak{p}}(\mathrm{Norm}(\mathfrak{p})^{-1/2}T) = 1 + a_1 T + \cdots + a_{2g-1}T^{2g-1} + T^{2g}.$$

This polynomial is determined by the point $(a_1, \dots, a_g)$ which lies in a bounded region of $\mathbb{R}^g$. It is natural to ask whether these points admit a limiting distribution as $\mathfrak{p}$ varies, and if so what this can be.

For E/K an elliptic curve, there are conjecturally 3 possible distributions, each corresponding to traces of random matrices:

- one when E has CM defined over K (matrices in $\mathrm{U}(1)$);
- one when E has CM not defined over K (matrices in $N(\mathrm{U}(1))$;
- one when E does not have CM (matrices in $\mathrm{SU}(2)$).

For illustrations, see https://math.mit.edu/~drew.

The Sato-Tate group of an abelian variety

Assume the **Mumford-Tate conjecture*** for A. Then there is a natural (but elaborate) construction of a compact Lie group $\mathrm{ST}(A)$ contained in $\mathrm{USp}(2g)$ and, for each $\mathfrak{p}$, a conjugacy class $\mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{ST}(A)$ with charpoly $\overline{L}_{\mathfrak{p}}(T)$. The **generalized Sato-Tate conjecture** is that the $\mathrm{Frob}_{\mathfrak{p}}$ are equidistributed with respect to (the image of) Haar measure.

This reduces to a statement about analytic continuation of the L-functions associated to irreducible representations of $\mathrm{ST}(A)$. This is known in certain cases, but we do not discuss this point here.

For $\dim(A) \leq 3$, $\mathrm{ST}(A)$ can be computed from the data of the $\mathbb{R}$-algebra $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}} := \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{R}$ and its $G_{\mathbb{Q}}$-action. This data can in principle be computed rigorously (Costa–Mascot–Sijsling–Voight).

*For any prime ℓ, the image of the ℓ-adic Galois representation of A has finite index in the maximal group allowed by the Hodge structure. This holds for $\dim(A) \leq 3$.

The connected and finite parts of the Sato-Tate group

There is a canonical exact sequence

$$1 \to \mathrm{ST}(A)^\circ \to \mathrm{ST}(A) \to \pi_0(\mathrm{ST}(A)) \to 1$$

where $\mathrm{ST}(A)^\circ$ is the identity component (and hence connected) and $\pi_0(\mathrm{ST}(A))$ is the component group (and hence finite).

The group $\mathrm{ST}(A)^\circ$ depends only on $A_{\overline{\mathbb{Q}}}$. It is equivalent data to the base change of the Mumford-Tate group (determined by the Hodge structure) from $\mathbb{Q}$ to $\overline{\mathbb{Q}}$.

The group $\pi_0(\mathrm{ST}(A))$ is the Galois group of a certain finite extension L/K. For $\dim(A) \leq 3$, L is the **endomorphism field** of A: the minimal extension for which $\mathrm{End}(A_L) = \mathrm{End}(A_{\overline{\mathbb{Q}}})$.

For example, if $\dim(A) = 1$ and A has CM by a quadratic field M not in K, then $L = MK$ and $\mathrm{ST}(A)/\mathrm{ST}(A)^\circ = N(\mathrm{U}(1))/\mathrm{U}(1) \cong \mathrm{Gal}(MK/K)$.

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The case of surfaces

Theorem (Fité–K–Rotger–Sutherland, 2012)

There are 52 *conjugacy classes of closed subgroups of* USp(4) *which occur as* ST(A) *for some abelian surface A over some number field K.*

- This includes 6 options for $\mathrm{ST}(A)^\circ$; see next slide.
- $\#\pi_0(\mathrm{ST}(A))$ divides $48 = 2^4 \times 3$ (and this value occurs).
- The 52 cases correspond to distinct distributions of $\overline{L}_\mathfrak{p}$.
- The theorem is quantified over all K. If we require $K = \mathbb{Q}$, then 34 cases occur. If we require K to be totally real, then 35 cases occur.
- There is a field K over which all 52 cases occur (Fité–Guitart).
- Nothing changes if we restrict to principally polarized abelian surfaces or even Jacobians. FKRS give explicit genus 2 curves in all cases.

Identity components vs. extensions: the case of surfaces

Type	$\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$	$\mathrm{ST}(A)^\circ$	Extensions	Maximal[†]
A	$\mathbb{R}$	$\mathrm{USp}(4)$	1	1
B	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	2	1
C	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	2	1
D	$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$	5	2
E	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$	10	2
F	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$	32	2
Total			52	9

Here $*_2$ denotes the diagonal embedding.

Warning: if A is geometrically simple, $\mathrm{ST}(A)^\circ$ can still be decomposable because it only depends on $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$. For example, if A has CM by a quartic field K, then $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \times \mathbb{C}$.

[†] Here "maximal" will always mean with respect to inclusions of **finite index**.

The case of threefolds

Theorem (Fité–K–Sutherland, 2019)

There are 410 conjugacy classes of closed subgroups of $\mathrm{USp}(6)$ *which occur as* $\mathrm{ST}(A)$ *for some abelian threefold* A *over some number field* K*.*

- This includes 14 options for $\mathrm{ST}(A)^\circ$ (Moonen–Zarhin).
- $\#\pi_0(\mathrm{ST}(A))$ divides[‡] one of $192 = 2^6 \times 3$, $336 = 2^4 \times 3 \times 7$, $432 = 2^4 \times 3^3$ (and these values occur).
- The 410 cases correspond to only 409 distinct distributions of $\overline{L}_{\mathfrak{p}}$. The two cases that collide have distinct component groups.
- We do not know what happens if we restrict K.
- It is unclear[§] if we can achieve a principal polarization or a Jacobian.

[‡]This refines earlier estimates by Silverberg and Guralnick-K.

[§]At AGCT 2019, I announced that every Sato-Tate group can be realized by a principally polarized abelian threefold. This claim is retracted.

Identity components vs. extensions: the case of threefolds

Type	$\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$	$\mathrm{ST}(A)^\circ$	Extensions	Maximal[¶]
A	$\mathbb{R}$	$\mathrm{USp}(6)$	1	1
B	$\mathbb{C}$	$\mathrm{U}(3)$	2	1
C	$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{USp}(4)$	1	1
D	$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{USp}(4)$	2	1
E	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	4	1
F	$\mathbb{C} \times \mathbb{R} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	5	1
G	$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$	5	2
H	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	13	3
I	$\mathbb{R} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	10	2
J	$\mathbb{R} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$	32	2
K	$\mathbb{C} \times \mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$	31	2
L	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1) \times \mathrm{U}(1)_2$	122	2
M	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$	11	2
N	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$	171	12
Total			410	33

[¶] Again, "maximal" means with respect to inclusions of finite index.

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Indecomposable cases: type **A**

For each candidate G° for $\mathrm{ST}(A)^\circ$, candidates for G correspond to conjugacy classes of finite subgroups of N/G° where N is the normalizer of G° in $\mathrm{USp}(6)$. We identify maximal candidates for G and describe realizations‖ of these by abelian threefolds over $\mathbb{Q}$. In most cases, these will not be Jacobians; more on that later.

To begin, we say that G° is **indecomposable** if $G^\circ = \mathrm{USp}(6), \mathrm{U}(3)$. In these cases, the only options for G are $\mathrm{USp}(6), \mathrm{U}(3), N(\mathrm{U}(3))$.

The group $\mathrm{USp}(6)$ occurs for an abelian threefold with trivial endomorphism ring. By a theorem of Zarhin, the Jacobian of

$$y^2 = x^7 - x + 1$$

is such an example over $\mathbb{Q}$.

‖ To be clear, we are not attempting to classify **all** possible realizations.

Indecomposable cases: type **B**

The group $N(\mathrm{U}(3))$ occurs for the Jacobian of a generic Picard curve

$$y^3 = P_4(x).$$

To take a concrete example, consider

$$y^3 = x^4 + x + 1.$$

It was shown by Upton that its Jacobian has maximal mod-67 Galois image, by computing Frobenius elements.

One can also show that it has maximal mod-2 Galois image by computing the action of Galois on the 27 odd theta characteristics corresponding to finite bitangents (the 28th being the line at infinity).

Split products: types **C**, **D**, **F**, **G**, **I**, **J**, **K**, **L**

We say that G° is a **split product** if it factors as a nontrivial product $G_1^\circ \times G_2^\circ$ with no shared factors between the two sides. That is,

$$G^\circ = \mathrm{SU}(2) \times \mathrm{USp}(4), \mathrm{U}(1) \times \mathrm{USp}(4),$$
$$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2),$$
$$\mathrm{SU}(2) \times \mathrm{SU}(2)_2, \mathrm{SU}(2) \times \mathrm{U}(1)_2, \mathrm{U}(1) \times \mathrm{SU}(2)_2, \mathrm{U}(1) \times \mathrm{U}(1)_2.$$

In these cases, N splits as $N_1 \times N_2$, so the same holds for maximal candidates for G.

By taking products of lower-dimensional examples, we see immediately that all maximal candidates for G arise for principally polarized abelian threefolds over $\mathbb{Q}$.

Triple products: types **E**, **H**

We say that G° is a **triple product** if it is a product of three copies of the same group. In these cases, N/G° is finite.

For $G^\circ = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, we have $N/G^\circ \cong \mathrm{S}_3$. This is realized by the Weil restriction of a non-CM elliptic curve over a non-Galois cubic field.

For $G^\circ = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$, the maximal permissible** candidates for G have

$$G/G^\circ \cong \mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2, \mathrm{C}_2 \times \mathrm{C}_4, \mathrm{C}_6.$$

The first two cases arise as products. The third occurs for the Jacobian of $y^2 = x^7 - 1$ or $y^3 = x^4 - x$.

**Here we account for restrictions imposed by Shimura's theory of CM types.

Triple diagonals: types **M**, **N**

We say that G° is a **triple diagonal** if $G^\circ = \mathrm{SU}(2)_3, \mathrm{U}(1)_3$. In these cases, N/G° is infinite, but there is a bound on the order of elements in N/G° coming from the **rationality condition** (not discussed here).

These cases may be realized in a uniform way: start with an abelian threefold A isogenous to the cube of an elliptic curve E with $\mathrm{End}(E)_{\mathbb{Q}} = M$, where M is either $\mathbb{Q}$ or an imaginary quadratic field of class number 1. Let

$$\rho \in H^1(G_{\mathbb{Q}}, \mathrm{Aut}(A_{\overline{\mathbb{Q}}}))$$

be a cocycle; we may then form a twist of A over $\mathbb{Q}$ whose Sato-Tate group is the projective image of ρ.

One obtains a polarization on the twist by averaging, but it need not be principal.

More on type **N**

The 12 maximal groups of type **N** are mostly realized as complex reflection groups within $\mathrm{GL}_3(M) \rtimes \mathrm{Gal}(M/\mathbb{Q})$. This makes it easy to make Galois cocycles, as the invariant ring of a complex reflection group is a polynomial ring (and conversely!).

One can also find explicit Galois embeddings using various techniques. Most notably, one of the groups arises from the Hessian group of order 216; this can be achieved using the mod-2 Galois representation of a generic Picard curve (which again can be computed from bitangents).

Scoreboard

Type	G°	Maximal	PPA3s††	Jacobians
~~**A**~~	~~$\mathrm{USp}(6)$~~	~~1~~	~~1~~	~~1~~
~~**B**~~	~~$\mathrm{U}(3)$~~	~~1~~	~~1~~	~~1~~
C	$\mathrm{SU}(2) \times \mathrm{USp}(4)$	1	1	0
D	$\mathrm{U}(1) \times \mathrm{USp}(4)$	1	1	0
E	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	1	1	0
F	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	1	1	0
G	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$	2	2	0
H	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	3	3	1
I	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	2	2	0
J	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$	2	2	0
K	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$	2	2	0
L	$\mathrm{U}(1) \times \mathrm{U}(1)_2$	2	2	0
M	$\mathrm{SU}(2)_3$	2	0	0
N	$\mathrm{U}(1)_3$	12	0	0

††PPA3 = principally polarized abelian threefold.

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Glueing along 2-torsion

One way to convert product constructions into Jacobians is to glue along 2-torsion. Let k be any field of characteristic 0.

Theorem (Howe–Leprevost–Poonen)

Let C_1, C_2, C_3 be curves of genus 1 over k. Suppose that $\prod_i \Delta(\mathrm{Jac}(C_i))$ is a square in k. Then there exist a curve C of genus 3 over k, a twist A of $\mathrm{Jac}(C)$ over k, and a $(2,2,2)$-isogeny $\mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2) \times \mathrm{Jac}(C_3) \cong A$.

Theorem (Hanselman)

Let C_1 and C_2 be curves of genus 1 and 2 over k. Suppose that there exist a point $Q \in \mathrm{Jac}(C_2)[2](k)$ and an isomorphism $\mathrm{Jac}(C_1)[2] \cong \langle Q \rangle^\perp / \langle Q \rangle$ of group schemes over k. Then there exist a curve C of genus 3 over k, a twist A of $\mathrm{Jac}(C)$ over k, and a $(2,2)$-isogeny $\mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2) \to A$.

In both cases, C and A can be constructed explicitly.

An example of type **H** (Everett Howe)

Consider the following elliptic curves over $\mathbb{Q}$.

$$\begin{aligned} E_1 : y^2 &= x^3 + 3x^2 + 3x && \text{CM by } \mathbb{Q}(\zeta_3) \\ E_2 : y^2 &= x^3 + x^2 + 2x && \text{CM by } \mathbb{Q}(\sqrt{-2}) \\ E_3 : y^2 &= x^3 - 21x && \text{CM by } \mathbb{Q}(i) \end{aligned}$$

Then $E_1 \times E_2 \times E_3$ is isogenous to a twist of the Jacobian of

$$3X^4 + 2Y^4 + 6Z^4 - 6X^2Y^2 + 6X^2Z^2 - 12Y^2Z^2 = 0.$$

This realizes a maximal extension of $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ with component group $\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2$.

We may similarly deal with the extension of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ by S_3 using conjugate curves over a cubic field.

More examples of glueing along 2-torsion

For any polynomial $P_4(x)$ of degree 4 with distinct roots, for

$$C_1 : y^2 = P_4(x), \quad C_2 : y^2 = xP_4(x), \quad C_3 : y^2 = P_4(x^2),$$

there is a $(2,2)$-isogeny $\mathrm{Jac}(C_1) \times \mathrm{Jac}(C_2) \to \mathrm{Jac}(C_3)$.

Type	G/G°	$P_4(x)$	$\mathrm{Jac}(C_1)$	CM
C	C_1	$4x^4 - 7x + 4$	2836.a1	None
D	C_2	$x^4 + 6x^2 + 4x + 2$	1600.m1	$\mathbb{Q}(i)$
F	$\mathrm{C}_2 \times \mathrm{C}_2$	$x^4 + 2x^3 + 4x^2 + 4x + 4$	256.a1	$\mathbb{Q}(i)$
G	C_4	$x^4 - 5x^3 + 10x^2 + 10x - 1$	200.b2	None
G	$\mathrm{C}_2 \times \mathrm{C}_2$	$x^4 + 4x^3 - 2x^2 + 4x + 1$	128.a2	None
H	$\mathrm{C}_2 \times \mathrm{C}_4$	$x^4 - 8x^3 + 20x^2 - 16x + 2$	256.d1	$\mathbb{Q}(\sqrt{-2})$
I	D_4	$x^4 + x^2 + 2$	224.a1	None
I	D_6	$x^4 + 8x^3 + 18x^2 + 16x - 4$	5184.d2	None
K	$\mathrm{C}_2 \times \mathrm{D}_4$	$x^4 + 2x^3 + 4x - 4$	2304.c2	$\mathbb{Q}(i)$

Scoreboard

Type	G°	Maximal	PPA3s	Jacobians
~~**A**~~	~~USp(6)~~	~~1~~	~~1~~	~~1~~
~~**B**~~	~~U(3)~~	~~1~~	~~1~~	~~1~~
~~**C**~~	~~$\mathrm{SU}(2) \times \mathrm{USp}(4)$~~	~~1~~	~~1~~	~~1~~
~~**D**~~	~~$\mathrm{U}(1) \times \mathrm{USp}(4)$~~	~~1~~	~~1~~	~~1~~
~~**E**~~	~~$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$~~	~~1~~	~~1~~	~~1~~
~~**F**~~	~~$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$~~	~~1~~	~~1~~	~~1~~
~~**G**~~	~~$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$~~	~~2~~	~~2~~	~~2~~
~~**H**~~	~~$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$~~	~~3~~	~~3~~	~~3~~
~~**I**~~	~~$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$~~	~~2~~	~~2~~	~~2~~
J	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$	2	2	0
K	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$	2	2	1
L	$\mathrm{U}(1) \times \mathrm{U}(1)_2$	2	0	0
M	$\mathrm{SU}(2)_3$	2	0	0
N	$\mathrm{U}(1)_3$	12	0	0

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Reduced automorphism groups of genus 3 curves

$\mathrm{Aut}'(C)$	Hyperelliptic model	Plane quartic model	Type
C_1	$P_8(x)$	$P_4(X,Y,Z)$	**A**
C_2	$P_4(x^2)$	$Y^4 + Y^2P_2(X,Z) + P_4(X,Z)$	**C**
C_3	None	$Y^3Z + P_4(X,Z)$	**B**
$\mathrm{C}_2 \times \mathrm{C}_2$	$x^4P_2(x^2 + x^{-2})$	$P_2(X^2, Y^2, Z^2)$	**E**
C_6	None	$Y^3Z + P_2(X^2, Z^2)$	**K**
C_7	$x^7 - 1\star$	None	**H**
C_9	None	$Y^3Z + X^4 + XZ^3\star$	**H**
S_3	$xP_2(x^3)$	$X^3Z + Y^3Z + aX^2Y^2 + bXYZ^2 + cZ^4$	**I**
D_4	$P_2(x^4)$	$\mathrm{Cyc}(X^4) + aX^2Y^2 + b(X^2+Y^2)Z^2$	**I**
D_6	$x^7 - x\star$	None	**N**
D_8	$x^8 - 1\star$	None	**L**
$\langle 16, 13\rangle$	None	$Y^4 + P_2(X^2, Z^2)$	**L**
S_4	$x^8 - 14x^4 + 1\star$	$\mathrm{Cyc}(X^4) + c\mathrm{Cyc}(X^2Y^2)$	**M**
$\langle 48, 33\rangle$	None	$Y^4 + X^3Z + Z^4\star$	**L**
$\langle 96, 64\rangle$	None	$X^4 + Y^4 + Z^4\star$	**N**
$\mathrm{PSL}_2(\mathbb{F}_7)$	None	$X^3Y + XZ^3 + Y^3Z\star$	**N**

Notation: Aut$'$: divide by the hyperelliptic involution; P_n: a generic polynomial (or homogeneous polynomials) of degree n; Cyc: sum over cyclic permutations of X, Y, Z; $\star$: an isolated point in moduli; $\langle m, n\rangle$: GAP group notation.

Twists of maximal automorphism groups

The curve

$$y^2 = x^8 - 14x^4 + 1$$

has reduced automorphism group S_4. Twists of this curve were studied by Arora–Cantoral Farfán–Landesman–Lombardo–Morrow; they give examples with $G^\circ = \mathrm{SU}(2)_3$ and $G/G^\circ \cong \mathrm{S}_4$.

Similarly, the curve

$$y^2 = x^7 - x$$

has reduced automorphism group D_6. Twists of this curve[‡‡] give examples with $G^\circ = \mathrm{U}(1)_3$ and $G/G^\circ \cong \langle 48, 38 \rangle$.

Similarly, twists of the Fermat and Klein quartics were studied by Fité–Lorenzo García–Sutherland; they give examples with $G^\circ = \mathrm{U}(1)_3$ and $G/G^\circ \cong \langle 192, 956 \rangle, \langle 336, 208 \rangle$.

[‡‡]Beware: twists of the form $y^2 = x^7 - cx$ are not general enough for this statement.

Twists of smaller automorphism groups

Families of curves which acquire various automorphism groups over $\overline{\mathbb{Q}}$ have been catalogued by Lorenzo García. For instance, curves of the form

$$X^3Z + bY^3Z + cX^2Y^2 + dXYZ^2 + eZ^4$$

admit automorphisms by S_3 over $\mathbb{Q}(\zeta_3, b^{1/3})$. They generally have Sato-Tate groups with $G^\circ \cong \mathrm{SU}(2) \times \mathrm{SU}(2)_2$; by specializing, we can ensure that $G^\circ \cong \mathrm{SU}(2)_3$ and $G/G^\circ \cong \mathrm{D}_6$.

For another example, curves of the form

$$Y^4 = P_4(X, Z)$$

admit automorphisms by D_4 over $L(i)$ where L is the splitting field of P_4. A generic such curve has $G^\circ \cong \mathrm{SU}(2) \times \mathrm{U}(1)_2$ and $G/G^\circ \cong \mathrm{S}_4 \times \mathrm{C}_2$ (that is, $G \cong \mathrm{SU}(2) \times J(O)$)).

Plane quartics with an involution

A plane quartic with an involution is a double cover of a genus-1 curve. The Prym is isomorphic (without polarization) to the Jacobian of a (possibly decomposable) genus-2 curve. (This includes our prior $1+2$ glueing.)

For example, the curve cut out by

$$4X^4 + 8X^3Z - 8X^2Y^2 + 6X^2Z^2 + 4XY^2Z - 4XZ^3 + 6Y^4 + 4Y^2Z^2 - 5Z^4$$

has Sato-Tate group $\mathrm{SU}(2) \times J(D_6)$.

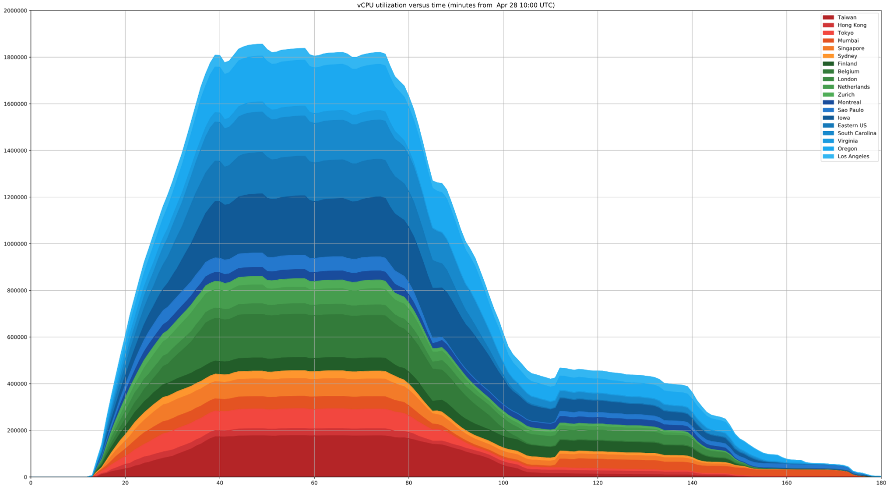
For another example, the curve cut out by

$$2Y^4 + Y^2(4X^2 - 6Z^2) + 4X^4 + 6X^3Z + XZ^3 + 3Z^4$$

has Sato-Tate group $N(\mathrm{U}(1)) \times J(E_6)$.

A word from our sponsors

Some of the preceding examples were first discovered using **very** large searches using Google Cloud Platform.



Scoreboard

Type	G°	Maximal	PPA3s	Jacobians
~~**A**~~	~~$USp(6)$~~	~~1~~	~~1~~	~~1~~
~~**B**~~	~~$U(3)$~~	~~1~~	~~1~~	~~1~~
~~**C**~~	~~$SU(2) \times USp(4)$~~	~~1~~	~~1~~	~~1~~
~~**D**~~	~~$U(1) \times USp(4)$~~	~~1~~	~~1~~	~~1~~
~~**E**~~	~~$SU(2) \times SU(2) \times SU(2)$~~	~~1~~	~~1~~	~~1~~
~~**F**~~	~~$U(1) \times SU(2) \times SU(2)$~~	~~1~~	~~1~~	~~1~~
~~**G**~~	~~$U(1) \times U(1) \times SU(2)$~~	~~2~~	~~2~~	~~2~~
~~**H**~~	~~$U(1) \times U(1) \times U(1)$~~	~~3~~	~~3~~	~~3~~
~~**I**~~	~~$SU(2) \times SU(2)_2$~~	~~2~~	~~2~~	~~2~~
~~**J**~~	~~$SU(2) \times U(1)_2$~~	~~2~~	~~2~~	~~2~~
~~**K**~~	~~$U(1) \times SU(2)_2$~~	~~2~~	~~2~~	~~2~~
L	$U(1) \times U(1)_2$	2	2	0
~~**M**~~	~~$SU(2)_3$~~	~~2~~	~~2~~	~~2~~
N	$U(1)_3$	12	3	3

Contents

The situation in type **L**

The two maximal groups of type **L** arise from the product of a CM elliptic curve with the Jacobian of a genus 2 curve with a maximal Sato-Tate group ($J(D_6)$ or $J(O)$).

It is possible to arrange for pairs of curves like this to admit a $(2,2)$-glueing, but there is a catch: this (apparently) forces the CM field of the elliptic curve to lie within the endomorphism field of the genus 2 Jacobian. So we miss the desired maximal groups by a factor of 2.

Can one use a $(3,3)$-glueing instead? If not, can one exhibit an obstruction to realizing these groups with Jacobians?

A closer look at type **N**

The 12 maximal groups of type **N** have component groups of the form $H \rtimes \mathrm{Gal}(M/\mathbb{Q})$ where M is an imaginary quadratic field of class number 1 and H is a finite subgroup of $\mathrm{GL}_3(M)$. We have realized three of these groups using Jacobians so far.

We get a fourth group by taking the product of an elliptic curve with CM by $\mathbb{Q}(\sqrt{-2})$ with the Jacobian of a genus 2 curve with Sato-Tate group $J(O)$.

We get a fifth group by taking the Weil restriction of $y^2 = x^3 - \alpha$ where $\mathbb{Q}(\alpha)$ is an S_3-cubic field.

The scoreboard in type **N**

H	$H \rtimes \mathrm{Gal}(M/\mathbb{Q})$	M	Realized as PPA3?
$\langle 24, 1\rangle$	$\langle 48, 15\rangle$	$\mathbb{Q}(i)$	no
$\langle 24, 10\rangle$	$\langle 48, 15\rangle$	$\mathbb{Q}(\zeta_3)$	no
$\langle 24, 5\rangle$	$\langle 48, 38\rangle$	$\mathbb{Q}(i)$	as Jacobian
$\langle 24, 5\rangle$	$\langle 48, 41\rangle$	$\mathbb{Q}(i)$	no
$\langle 48, 29\rangle$	$\langle 96, 193\rangle$	$\mathbb{Q}(\sqrt{-2})$	as product
$\langle 72, 25\rangle$	$\langle 144, 127\rangle$	$\mathbb{Q}(\zeta_3)$	no
$\langle 72, 25\rangle$	$\langle 144, 125\rangle$	$\mathbb{Q}(\zeta_3)$	no
$\langle 96, 67\rangle$	$\langle 192, 988\rangle$	$\mathbb{Q}(i)$	no
$\langle 96, 64\rangle$	$\langle 192, 956\rangle$	$\mathbb{Q}(i)$	as Jacobian
$\langle 168, 42\rangle$	$\langle 336, 208\rangle$	$\mathbb{Q}(\sqrt{-7})$	as Jacobian
$\langle 216, 92\rangle$	$\langle 432, 523\rangle$	$\mathbb{Q}(\zeta_3)$	as Weil restriction
$\langle 216, 153\rangle$	$\langle 432, 734\rangle$	$\mathbb{Q}(\zeta_3)$	no

The remaining cases

It is unclear whether the remaining cases in type **N** can occur for PPA3s. It may be possible to rule this out by carefully classifying the ways they can occur, in the style of Fité–Guitart; a first step would be show that one **must** take a twist of an abelian threefold isogenous to the cube of an elliptic curve with CM by M.

Regardless, one can still look for explicit realizations, say by taking the product of an elliptic curve with the Prym variety of dimension 2 associated to a ramified double or triple cover; this would yield a polarization of type $(1, 1, 2)$ or $(1, 1, 3)$. For example, $\langle 192, 988 \rangle$ occurs in this manner (for a double cover).

The final scoreboard (for now)

Type	G°	Maximal	PPA3s	Jacobians
A	USp(6)	1	1	1
B	U(3)	1	1	1
C	$\mathrm{SU}(2) \times \mathrm{USp}(4)$	1	1	1
D	$\mathrm{U}(1) \times \mathrm{USp}(4)$	1	1	1
E	$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	1	1	1
F	$\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	1	1	1
G	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$	2	2	2
H	$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	3	3	3
I	$\mathrm{SU}(2) \times \mathrm{SU}(2)_2$	2	2	2
J	$\mathrm{SU}(2) \times \mathrm{U}(1)_2$	2	2	2
K	$\mathrm{U}(1) \times \mathrm{SU}(2)_2$	2	2	2
L	$\mathrm{U}(1) \times \mathrm{U}(1)_2$	2	2	0
M	$\mathrm{SU}(2)_3$	2	2	2
N	$\mathrm{U}(1)_3$	12	5	3