Towards explicit realizations of the 
Sato-Tate groups of abelian threefolds

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joint work (in progress) with Francesc Fité and Andrew V. Sutherland

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Around Frobenius distributions and related topics (virtual conference)
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6. Possible obstructions to going further
Zeta functions of algebraic varieties

For $X$ an algebraic variety over a finite field $\mathbb{F}_q$, the zeta function of $X$ is

$$Z(X, T) = \prod_{x \in X^\circ} (1 - T^{\deg(x/\mathbb{F}_q)})^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_q^n)\right),$$

where $X^\circ$ denotes the closed points of $X$ (i.e., Galois orbits of $\overline{\mathbb{F}}_q$-points).

For $X$ smooth proper over $\mathbb{F}_q$, we have

$$Z(X, T) = \frac{P_1(T) \cdots P_{2g-1}(T)}{P_0(T) \cdots P_{2g}(T)}$$

where $P_i(T)$ is (the reverse of) a $q^i$-Weil polynomial:

- $P_i(T)$ has integer coefficients and its constant term is 1.
- The roots of $P_i(T)$ in $\mathbb{C}$ all lie on the circle $|T| = q^{-i/2}$. 
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Curves and abelian varieties

When $X$ is a (smooth, proper, geometrically integral) curve of genus $g$,

$$P_0(T) = 1 - T, \quad P_2(T) = 1 - qT,$$

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When $X$ is an abelian variety of dimension $g$, $P_1(T)$ is of degree $2g$, $P_1(q^{-1/2}T)$ is palindromic, and $P_i(T) = \wedge^i P_1(T)$. That is, if $P_1$ has roots $\alpha_1, \ldots, \alpha_{2g}$, then $P_i$ has roots

$$\alpha_{j_1} \cdots \alpha_{j_i} \quad (1 \leq j_1 < \cdots < j_i \leq 2g).$$

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L-functions

For $A$ an abelian variety over a number field $K$ with ring of integers $\mathfrak{o}_K$, its (incomplete) $L$-function is the Dirichlet series

$$L(A, s) = \prod_{p} L_p(\text{Norm}(p)^{-s})^{-1}$$

where $p$ runs over prime ideals of $\mathfrak{o}_K$ at which $A$ has good reduction, $\text{Norm}(p) = \#(\mathfrak{o}_K/p)$ is the absolute norm, and $L_p(T)$ is the factor $P_1(T)$ of the zeta function of the reduction of $A$ modulo $p$.

For example, if $A$ is an elliptic curve over $\mathbb{Q}$, this is the usual expression

$$L(A, s) = \prod_{p}(1 - a_p p^{-s} + p^{1-2s})^{-1}, \quad a_p = p + 1 - \#A(\mathbb{F}_p).$$

In general, $L(A, s)$ converges absolutely for $\text{Re}(s) > 3/2$ but is expected to admit a meromorphic continuation to $\mathbb{C}$. 
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Distribution of Euler factors

With the functional equation in mind, we renormalize the $L$-polynomials:

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\overline{L}_p(T) = L_p(\text{Norm}(p)^{-1/2} T) = 1 + a_1 T + \cdots + a_{2g-1} T^{2g-1} + T^{2g}.
$$

This polynomial is determined by the point $(a_1, \ldots, a_g)$ which lies in a bounded region of $\mathbb{R}^g$. It is natural to ask whether these points admit a limiting distribution as $p$ varies, and if so what this can be.

For $E/K$ an elliptic curve, there are conjecturally 3 possible distributions, each corresponding to traces of random matrices:

- one when $E$ has CM defined over $K$ (matrices in $U(1)$);
- one when $E$ has CM not defined over $K$ (matrices in $N(U(1))$;
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For illustrations, see https://math.mit.edu/~drew.
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Assume the **Mumford-Tate conjecture*** for \( A \). Then there is a natural (but elaborate) construction of a compact Lie group \( ST(A) \) contained in \( USp(2g) \) and, for each \( p \), a conjugacy class \( \text{Frob}_p \) in \( ST(A) \) with charpoly \( L_p(T) \). The **generalized Sato-Tate conjecture** is that the \( \text{Frob}_p \) are equidistributed with respect to (the image of) Haar measure.

This reduces to a statement about analytic continuation of the \( L \)-functions associated to irreducible representations of \( ST(A) \). This is known in certain cases, but we do not discuss this point here.

For \( \dim(A) \leq 3 \), \( ST(A) \) can be computed from the data of the \( \mathbb{R} \)-algebra \( \text{End}(A_{\mathbb{Q}})_{\mathbb{R}} := \text{End}(A_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{R} \) and its \( G_{\mathbb{Q}} \)-action. This data can in principle be computed rigorously (Costa–Mascot–Sijsling–Voight).

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*For any prime \( \ell \), the image of the \( \ell \)-adic Galois representation of \( A \) has finite index in the maximal group allowed by the Hodge structure. This holds for \( \dim(A) \leq 3 \).
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The connected and finite parts of the Sato-Tate group

There is a canonical exact sequence

\[ 1 \rightarrow \text{ST}(A) \rightarrow \text{ST}(A) \rightarrow \pi_0(\text{ST}(A)) \rightarrow 1 \]

where \( \text{ST}(A) \) is the identity component (and hence connected) and \( \pi_0(\text{ST}(A)) \) is the component group (and hence finite).

The group \( \text{ST}(A) \) depends only on \( A_{\overline{\mathbb{Q}}} \). It is equivalent data to the base change of the Mumford-Tate group (determined by the Hodge structure) from \( \mathbb{Q} \) to \( \overline{\mathbb{Q}} \).

The group \( \pi_0(\text{ST}(A)) \) is the Galois group of a certain finite extension \( L/K \). For \( \dim(A) \leq 3 \), \( L \) is the endomorphism field of \( A \): the minimal extension for which \( \text{End}(A_L) = \text{End}(A_{\overline{\mathbb{Q}}}) \).

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6 Possible obstructions to going further
The case of surfaces

Theorem (Fité–K–Rotger–Sutherland, 2012)

There are 52 conjugacy classes of closed subgroups of $\text{USp}(4)$ which occur as $\text{ST}(A)$ for some abelian surface $A$ over some number field $K$.

- This includes 6 options for $\text{ST}(A)^\circ$; see next slide.
- $\#\pi_0(\text{ST}(A))$ divides $48 = 2^4 \times 3$ (and this value occurs).
- The 52 cases correspond to distinct distributions of $\overline{L}_p$.
- The theorem is quantified over all $K$. If we require $K = \mathbb{Q}$, then 34 cases occur. If we require $K$ to be totally real, then 35 cases occur.
- There is a field $K$ over which all 52 cases occur (Fité–Guitart).
- Nothing changes if we restrict to principally polarized abelian surfaces or even Jacobians. FKRS give explicit genus 2 curves in all cases.
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There are 52 conjugacy classes of closed subgroups of $\text{USp}(4)$ which occur as $\text{ST}(A)$ for some abelian surface $A$ over some number field $K$.

- This includes 6 options for $\text{ST}(A)^\circ$; see next slide.
- $\#\pi_0(\text{ST}(A))$ divides $48 = 2^4 \times 3$ (and this value occurs).
- The 52 cases correspond to distinct distributions of $\overline{L}_p$.
- The theorem is quantified over all $K$. If we require $K = \mathbb{Q}$, then 34 cases occur. If we require $K$ to be totally real, then 35 cases occur.
- There is a field $K$ over which all 52 cases occur (Fité–Guitart).
- Nothing changes if we restrict to principally polarized abelian surfaces or even Jacobians. FKRS give explicit genus 2 curves in all cases.
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### Identity components vs. extensions: the case of surfaces

<table>
<thead>
<tr>
<th>Type</th>
<th>( \text{End}(A_{\mathbb{Q}})_{\mathbb{R}} )</th>
<th>( \text{ST}(A)^{\circ} )</th>
<th>Extensions</th>
<th>Maximal†</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \mathbb{R} )</td>
<td>( \text{USp}(4) )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>( \mathbb{R} \times \mathbb{R} )</td>
<td>( \text{SU}(2) \times \text{SU}(2) )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>( \mathbb{C} \times \mathbb{R} )</td>
<td>( \text{U}(1) \times \text{SU}(2) )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>( \mathbb{C} \times \mathbb{C} )</td>
<td>( \text{U}(1) \times \text{U}(1) )</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>( M_2(\mathbb{R}) )</td>
<td>( \text{SU}(2)_2 )</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>F</td>
<td>( M_2(\mathbb{C}) )</td>
<td>( \text{U}(1)_2 )</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>52</td>
<td>9</td>
</tr>
</tbody>
</table>

Here \( \ast_2 \) denotes the diagonal embedding.

**Warning:** if \( A \) is geometrically simple, \( \text{ST}(A)^{\circ} \) can still be decomposable because it only depends on \( \text{End}(A_{\mathbb{Q}})_{\mathbb{R}} \). For example, if \( A \) has CM by a quartic field \( K \), then \( \text{End}(A_{\mathbb{Q}})_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \times \mathbb{C} \).

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<td>USp(4)</td>
<td>1</td>
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</tr>
<tr>
<td>B</td>
<td>( \mathbb{R} \times \mathbb{R} )</td>
<td>SU(2) \times SU(2)</td>
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There are 410 conjugacy classes of closed subgroups of USp(6) which occur as ST(A) for some abelian threefold A over some number field K.

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| Type | $\text{End}(A_{\mathbb{Q}})_{\mathbb{R}}$ | $\text{ST}(A)^{\circ}$ | Extensions | Maximal
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\mathbb{R}$</td>
<td>$\text{USp}(6)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>$\mathbb{C}$</td>
<td>$\text{U}(3)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>$\mathbb{R} \times \mathbb{R}$</td>
<td>$\text{SU}(2) \times \text{USp}(4)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>$\mathbb{C} \times \mathbb{R}$</td>
<td>$\text{U}(1) \times \text{USp}(4)$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>E</td>
<td>$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$</td>
<td>$\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$</td>
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<tr>
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<td>$\text{U}(1) \times \text{SU}(2) \times \text{SU}(2)$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>G</td>
<td>$\mathbb{C} \times \mathbb{C} \times \mathbb{R}$</td>
<td>$\text{U}(1) \times \text{U}(1) \times \text{SU}(2)$</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>H</td>
<td>$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$</td>
<td>$\text{U}(1) \times \text{U}(1) \times \text{U}(1)$</td>
<td>13</td>
<td>3</td>
</tr>
<tr>
<td>I</td>
<td>$\mathbb{R} \times \mathbb{M}_2(\mathbb{R})$</td>
<td>$\text{SU}(2) \times \text{SU}(2)_2$</td>
<td>10</td>
<td>2</td>
</tr>
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<td>J</td>
<td>$\mathbb{R} \times \mathbb{M}_2(\mathbb{C})$</td>
<td>$\text{SU}(2) \times \text{U}(1)_2$</td>
<td>32</td>
<td>2</td>
</tr>
<tr>
<td>K</td>
<td>$\mathbb{C} \times \mathbb{M}_2(\mathbb{R})$</td>
<td>$\text{U}(1) \times \text{SU}(2)_2$</td>
<td>31</td>
<td>2</td>
</tr>
<tr>
<td>L</td>
<td>$\mathbb{C} \times \mathbb{M}_2(\mathbb{C})$</td>
<td>$\text{U}(1) \times \text{U}(1)_2$</td>
<td>122</td>
<td>2</td>
</tr>
<tr>
<td>M</td>
<td>$\mathbb{M}_3(\mathbb{R})$</td>
<td>$\text{SU}(2)_3$</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>N</td>
<td>$\mathbb{M}_3(\mathbb{C})$</td>
<td>$\text{U}(1)_3$</td>
<td>171</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>410</strong></td>
<td><strong>33</strong></td>
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Contents

1. The Sato-Tate group of an abelian variety
2. Sato-Tate groups of abelian surfaces and threefolds
3. Realization by abelian threefolds
4. Realization by Jacobians: glueing along 2-torsion
5. Realization by Jacobians: automorphisms
6. Possible obstructions to going further
Indecomposable cases: type A

For each candidate $G^\circ$ for $\text{ST}(A)^\circ$, candidates for $G$ correspond to conjugacy classes of finite subgroups of $N/G^\circ$ where $N$ is the normalizer of $G^\circ$ in $\text{USp}(6)$. We identify maximal candidates for $G$ and describe realizations of these by abelian threefolds over $\mathbb{Q}$. In most cases, these will not be Jacobians; more on that later.

To begin, we say that $G^\circ$ is indecomposable if $G^\circ = \text{USp}(6), \text{U}(3)$. In these cases, the only options for $G$ are $\text{USp}(6), \text{U}(3), N(\text{U}(3))$.

The group $\text{USp}(6)$ occurs for an abelian threefold with trivial endomorphism ring. By a theorem of Zarhin, the Jacobian of

$$y^2 = x^7 - x + 1$$

is such an example over $\mathbb{Q}$.


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\text{To be clear, we are not attempting to classify all possible realizations.}
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The group \( N(U(3)) \) occurs for the Jacobian of a generic Picard curve

\[ y^3 = P_4(x). \]

To take a concrete example, consider

\[ y^3 = x^4 + x + 1. \]

It was shown by Upton that its Jacobian has maximal mod-67 Galois image, by computing Frobenius elements.

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We say that $G^\circ$ is a **split product** if it factors as a nontrivial product $G_1^\circ \times G_2^\circ$ with no shared factors between the two sides. That is,

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\end{align*}$$

In these cases, $N$ splits as $N_1 \times N_2$, so the same holds for maximal candidates for $G$.

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For $G^\circ = U(1) \times U(1) \times U(1)$, the maximal permissible** candidates for $G$ have

$$G/G^\circ \cong C_2 \times C_2 \times C_2, C_2 \times C_4, C_6.$$ 

The first two cases arise as products. The third occurs for the Jacobian of $y^2 = x^7 - 1$ or $y^3 = x^4 - x$.

**Here we account for restrictions imposed by Shimura’s theory of CM types.
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For $G^\circ = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$, we have $N/G^\circ \cong S_3$. This is realized by the Weil restriction of a non-CM elliptic curve over a non-Galois cubic field.

For $G^\circ = \text{U}(1) \times \text{U}(1) \times \text{U}(1)$, the maximal permissible** candidates for $G$ have

$$G/G^\circ \cong C_2 \times C_2 \times C_2, C_2 \times C_4, C_6.$$  

The first two cases arise as products. The third occurs for the Jacobian of $y^2 = x^7 - 1$ or $y^3 = x^4 - x$.

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We say that $G^\circ$ is a **triple diagonal** if $G^\circ = \text{SU}(2)_3, \text{U}(1)_3$. In these cases, $N/G^\circ$ is infinite, but there is a bound on the order of elements in $N/G^\circ$ coming from the **rationality condition** (not discussed here).

These cases may be realized in a uniform way: start with an abelian threefold $A$ isogenous to the cube of an elliptic curve $E$ with $\text{End}(E)_\mathbb{Q} = M$, where $M$ is either $\mathbb{Q}$ or an imaginary quadratic field of class number 1. Let

$$\rho \in H^1(G_\mathbb{Q}, \text{Aut}(A_\mathbb{Q}))$$

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<tr>
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<th>Maximal</th>
<th>PPA3s(^{\dagger\dagger})</th>
<th>Jacobians</th>
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<tr>
<td>A</td>
<td>USp(6)</td>
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<tr>
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<td>N</td>
<td>U(1)(_3)</td>
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\(^{\dagger\dagger}\text{PPA3 = principally polarized abelian threefold.}\)
Contents

1. The Sato-Tate group of an abelian variety
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5. Realization by Jacobians: automorphisms
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Glueing along 2-torsion

One way to convert product constructions into Jacobians is to glue along 2-torsion. Let $k$ be any field of characteristic 0.

**Theorem (Howe–Leprevost–Poonen)**

Let $C_1, C_2, C_3$ be curves of genus 1 over $k$. Suppose that $\prod_i \Delta(Jac(C_i))$ is a square in $k$. Then there exist a curve $C$ of genus 3 over $k$, a twist $A$ of $Jac(C)$ over $k$, and a $(2, 2, 2)$-isogeny $Jac(C_1) \times Jac(C_2) \times Jac(C_3) \cong A$.

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Let $C_1$ and $C_2$ be curves of genus 1 and 2 over $k$. Suppose that there exist a point $Q \in Jac(C_2)[2](k)$ and an isomorphism $Jac(C_1)[2] \cong \langle Q \rangle^\perp / \langle Q \rangle$ of group schemes over $k$. Then there exist a curve $C$ of genus 3 over $k$, a twist $A$ of $Jac(C)$ over $k$, and a $(2, 2)$-isogeny $Jac(C_1) \times Jac(C_2) \to A$.

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Consider the following elliptic curves over $\mathbb{Q}$.

\[ E_1 : y^2 = x^3 + 3x^2 + 3x \quad \text{CM by } \mathbb{Q}(\zeta_3) \]
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Then $E_1 \times E_2 \times E_3$ is isogenous to a twist of the Jacobian of
\[ 3X^4 + 2Y^4 + 6Z^4 - 6X^2Y^2 + 6X^2Z^2 - 12Y^2Z^2 = 0. \]

This realizes a maximal extension of $U(1) \times U(1) \times U(1)$ with component group $C_2 \times C_2 \times C_2$.

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<tr>
<th>Type</th>
<th>$G/G^\circ$</th>
<th>$P_4(x)$</th>
<th>$\text{Jac}(C_1)$</th>
<th>CM</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>$C_1$</td>
<td>$4x^4 - 7x + 4$</td>
<td>2836.a1</td>
<td>None</td>
</tr>
<tr>
<td>D</td>
<td>$C_2$</td>
<td>$x^4 + 6x^2 + 4x + 2$</td>
<td>1600.m1</td>
<td>$\mathbb{Q}(i)$</td>
</tr>
<tr>
<td>F</td>
<td>$C_2 \times C_2$</td>
<td>$x^4 + 2x^3 + 4x^2 + 4x + 4$</td>
<td>256.a1</td>
<td>$\mathbb{Q}(i)$</td>
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<tr>
<td>G</td>
<td>$C_4$</td>
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<td>None</td>
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<tr>
<td>G</td>
<td>$C_2 \times C_2$</td>
<td>$x^4 + 4x^3 - 2x^2 + 4x + 1$</td>
<td>128.a2</td>
<td>None</td>
</tr>
<tr>
<td>H</td>
<td>$C_2 \times C_4$</td>
<td>$x^4 - 8x^3 + 20x^2 - 16x + 2$</td>
<td>256.d1</td>
<td>$\mathbb{Q}(-2)$</td>
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<tr>
<td>I</td>
<td>$D_4$</td>
<td>$x^4 + x^2 + 2$</td>
<td>224.a1</td>
<td>None</td>
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<tr>
<td>J</td>
<td>$D_6$</td>
<td>$x^4 + 8x^3 + 18x^2 + 16x - 4$</td>
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<tr>
<td>K</td>
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<td>$x^4 + 2x^3 + 4x - 4$</td>
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<tr>
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<tr>
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### Reduced automorphism groups of genus 3 curves

<table>
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<tr>
<th>Aut'($C$)</th>
<th>Hyperelliptic model</th>
<th>Plane quartic model</th>
<th>Type</th>
</tr>
</thead>
<tbody>
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<td>$P_8(x)$</td>
<td>$P_4(X, Y, Z)$</td>
<td>$A$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$P_4(x^2)$</td>
<td>$Y^4 + Y^2P_2(X, Z) + P_4(X, Z)$</td>
<td>$C$</td>
</tr>
<tr>
<td>$C_3$</td>
<td>None</td>
<td>$Y^3Z + P_4(X, Z)$</td>
<td>$B$</td>
</tr>
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<td>$C_2 \times C_2$</td>
<td>$x^4P_2(x^2 + x^{-2})$</td>
<td>$P_2(X^2, Y^2, Z^2)$</td>
<td>$E$</td>
</tr>
<tr>
<td>$C_6$</td>
<td>None</td>
<td>$Y^3Z + P_2(X^2, Z^2)$</td>
<td>$K$</td>
</tr>
<tr>
<td>$C_7$</td>
<td>$x^7 - 1^*$</td>
<td>None</td>
<td>$H$</td>
</tr>
<tr>
<td>$C_9$</td>
<td>None</td>
<td>$Y^3 Z + X^4 + XZ^3^*$</td>
<td>$H$</td>
</tr>
<tr>
<td>$S_3$</td>
<td>$xP_2(x^3)$</td>
<td>$X^3 Z + Y^3 Z + aX^2 Y^2 + bXYZ^2 + cZ^4$</td>
<td>$I$</td>
</tr>
<tr>
<td>$D_4$</td>
<td>$P_2(x^4)$</td>
<td>$\text{Cyc}(X^4) + aX^2 Y^2 + b(X^2 + Y^2)Z^2$</td>
<td>$I$</td>
</tr>
<tr>
<td>$D_6$</td>
<td>$x^7 - x^*$</td>
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<td>$N$</td>
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<tr>
<td>$D_8$</td>
<td>$x^8 - 1^*$</td>
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<tr>
<td>$\langle 16, 13 \rangle$</td>
<td>None</td>
<td>$Y^4 + P_2(X^2, Z^2)$</td>
<td>$L$</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$x^8 - 14x^4 + 1^*$</td>
<td>$\text{Cyc}(X^4) + c\text{Cyc}(X^2 Y^2)$</td>
<td>$M$</td>
</tr>
<tr>
<td>$\langle 48, 33 \rangle$</td>
<td>None</td>
<td>$Y^4 + X^3 Z + Z^4^*$</td>
<td>$L$</td>
</tr>
<tr>
<td>$\langle 96, 64 \rangle$</td>
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<td>$X^4 + Y^4 + Z^4^*$</td>
<td>$N$</td>
</tr>
<tr>
<td>PSL$_2(\mathbb{F}_7)$</td>
<td>None</td>
<td>$X^3 Y + XZ^3 + Y^3 Z^*$</td>
<td>$N$</td>
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</tbody>
</table>

Notation: Aut': divide by the hyperelliptic involution; $P_n$: a generic polynomial (or homogeneous polynomials) of degree $n$; Cyc: sum over cyclic permutations of $X, Y, Z$; $^*$: an isolated point in moduli; $\langle m, n \rangle$: GAP group notation.
Twists of maximal automorphism groups

The curve

\[ y^2 = x^8 - 14x^4 + 1 \]

has reduced automorphism group \( S_4 \). Twists of this curve were studied by Arora–Cantoral Farfán–Landesman–Lombardo–Morrow; they give examples with \( G^\circ = SU(2)_3 \) and \( G/G^\circ \cong S_4 \).

Similarly, the curve

\[ y^2 = x^7 - x \]

has reduced automorphism group \( D_6 \). Twists of this curve\(^\ddagger\ddagger\) give examples with \( G^\circ = U(1)_3 \) and \( G/G^\circ \cong \langle 48, 38 \rangle \).

Similarly, twists of the Fermat and Klein quartics were studied by Fité–Lorenzo García–Sutherland; they give examples with \( G^\circ = U(1)_3 \) and \( G/G^\circ \cong \langle 192, 956 \rangle, \langle 336, 208 \rangle \).

\(^\ddagger\ddagger\)Beware: twists of the form \( y^2 = x^7 - cx \) are not general enough for this statement.
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Twists of smaller automorphism groups

Families of curves which acquire various automorphism groups over $\overline{\mathbb{Q}}$ have been catalogued by Lorenzo García. For instance, curves of the form

$$X^3Z + bY^3Z + cX^2Y^2 + dXYZ^2 + eZ^4$$

admit automorphisms by $S_3$ over $\mathbb{Q}(\zeta_3, b^{1/3})$. They generally have Sato-Tate groups with $G^\circ \cong SU(2) \times SU(2)_2$; by specializing, we can ensure that $G^\circ \cong SU(2)_3$ and $G/G^\circ \cong D_6$.

For another example, curves of the form

$$Y^4 = P_4(X, Z)$$

admit automorphisms by $D_4$ over $L(i)$ where $L$ is the splitting field of $P_4$. A generic such curve has $G^\circ \cong SU(2) \times U(1)_2$ and $G/G^\circ \cong S_4 \times C_2$ (that is, $G \cong SU(2) \times J(O)$).
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Plane quartics with an involution

A plane quartic with an involution is a double cover of a genus-1 curve. The Prym is isomorphic (without polarization) to the Jacobian of a (possibly decomposable) genus-2 curve. (This includes our prior $1+2$ glueing.)

For example, the curve cut out by

$$4X^4 + 8X^3Z - 8X^2Y^2 + 6X^2Z^2 + 4XY^2Z - 4XZ^3 + 6Y^4 + 4Y^2Z^2 - 5Z^4$$

has Sato-Tate group $SU(2) \times J(D_6)$.

For another example, the curve cut out by

$$2Y^4 + Y^2(4X^2 - 6Z^2) + 4X^4 + 6X^3Z + XZ^3 + 3Z^4$$

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A word from our sponsors

Some of the preceding examples were first discovered using *very* large searches using Google Cloud Platform.
### Scoreboard

<table>
<thead>
<tr>
<th>Type</th>
<th>$G^\circ$</th>
<th>Maximal</th>
<th>PPA3s</th>
<th>Jacobians</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>USp(6)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>U(3)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>SU(2) $\times$ USp(4)</td>
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<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>U(1) $\times$ USp(4)</td>
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<tr>
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<td>1</td>
<td>1</td>
</tr>
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<td>2</td>
<td>2</td>
</tr>
<tr>
<td>J</td>
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<td>U(1) $\times$ SU(2) $\times$ SU(2)</td>
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<tr>
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<td>0</td>
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<tr>
<td>M</td>
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<td>2</td>
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<td>N</td>
<td>U(1) $\times$ U(1)</td>
<td>12</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
Possible obstructions to going further

Contents

1 The Sato-Tate group of an abelian variety
2 Sato-Tate groups of abelian surfaces and threefolds
3 Realization by abelian threefolds
4 Realization by Jacobians: glueing along 2-torsion
5 Realization by Jacobians: automorphisms
6 Possible obstructions to going further
The situation in type $L$

The two maximal groups of type $L$ arise from the product of a CM elliptic curve with the Jacobian of a genus 2 curve with a maximal Sato-Tate group ($J(D_6)$ or $J(O)$).

It is possible to arrange for pairs of curves like this to admit a (2, 2)-glueing, but there is a catch: this (apparently) forces the CM field of the elliptic curve to lie within the endomorphism field of the genus 2 Jacobian. So we miss the desired maximal groups by a factor of 2.

Can one use a (3, 3)-glueing instead? If not, can one exhibit an obstruction to realizing these groups with Jacobians?
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A closer look at type $\mathbf{N}$

The 12 maximal groups of type $\mathbf{N}$ have component groups of the form $H \rtimes \text{Gal}(M/\mathbb{Q})$ where $M$ is an imaginary quadratic field of class number 1 and $H$ is a finite subgroup of $\text{GL}_3(M)$. We have realized three of these groups using Jacobians so far.

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## The scoreboard in type $\mathbf{N}$

<table>
<thead>
<tr>
<th>$H$</th>
<th>$H \rtimes \text{Gal}(M/\mathbb{Q})$</th>
<th>$M$</th>
<th>Realized as PPA3?</th>
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</thead>
<tbody>
<tr>
<td>$\langle 24, 1 \rangle$</td>
<td>$\langle 48, 15 \rangle$</td>
<td>$\mathbb{Q}(i)$</td>
<td>no</td>
</tr>
<tr>
<td>$\langle 24, 10 \rangle$</td>
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<td>$\mathbb{Q}(\zeta_3)$</td>
<td>no</td>
</tr>
<tr>
<td>$\langle 24, 5 \rangle$</td>
<td>$\langle 48, 38 \rangle$</td>
<td>$\mathbb{Q}(i)$</td>
<td>as Jacobian</td>
</tr>
<tr>
<td>$\langle 24, 5 \rangle$</td>
<td>$\langle 48, 41 \rangle$</td>
<td>$\mathbb{Q}(i)$</td>
<td>no</td>
</tr>
<tr>
<td>$\langle 48, 29 \rangle$</td>
<td>$\langle 96, 193 \rangle$</td>
<td>$\mathbb{Q}(\sqrt{-2})$</td>
<td>as product</td>
</tr>
<tr>
<td>$\langle 72, 25 \rangle$</td>
<td>$\langle 144, 127 \rangle$</td>
<td>$\mathbb{Q}(\zeta_3)$</td>
<td>no</td>
</tr>
<tr>
<td>$\langle 72, 25 \rangle$</td>
<td>$\langle 144, 125 \rangle$</td>
<td>$\mathbb{Q}(\zeta_3)$</td>
<td>no</td>
</tr>
<tr>
<td>$\langle 96, 67 \rangle$</td>
<td>$\langle 192, 988 \rangle$</td>
<td>$\mathbb{Q}(i)$</td>
<td>no</td>
</tr>
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<td>$\langle 96, 64 \rangle$</td>
<td>$\langle 192, 956 \rangle$</td>
<td>$\mathbb{Q}(i)$</td>
<td>as Jacobian</td>
</tr>
<tr>
<td>$\langle 168, 42 \rangle$</td>
<td>$\langle 336, 208 \rangle$</td>
<td>$\mathbb{Q}(\sqrt{-7})$</td>
<td>as Jacobian</td>
</tr>
<tr>
<td>$\langle 216, 92 \rangle$</td>
<td>$\langle 432, 523 \rangle$</td>
<td>$\mathbb{Q}(\zeta_3)$</td>
<td>as Weil restriction</td>
</tr>
<tr>
<td>$\langle 216, 153 \rangle$</td>
<td>$\langle 432, 734 \rangle$</td>
<td>$\mathbb{Q}(\zeta_3)$</td>
<td>no</td>
</tr>
</tbody>
</table>
The remaining cases

It is unclear whether the remaining cases in type $\mathbf{N}$ can occur for PPA3s. It may be possible to rule this out by carefully classifying the ways they can occur, in the style of Fité–Guitart; a first step would be show that one must take a twist of an abelian threefold isogenous to the cube of an elliptic curve with CM by $M$.

Regardless, one can still look for explicit realizations, say by taking the product of an elliptic curve with the Prym variety of dimension 2 associated to a ramified double or triple cover; this would yield a polarization of type $(1, 1, 2)$ or $(1, 1, 3)$. For example, $\langle 192, 988 \rangle$ occurs in this manner (for a double cover).
The final scoreboard (for now)

<table>
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<td>2</td>
</tr>
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<td>M</td>
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<td>2</td>
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<tr>
<td>N</td>
<td>U(1)$_3$</td>
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<td>5</td>
<td>3</td>
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</table>