Relative (φ, Γ)-modules

Kiran S. Kedlaya

Department of Mathematics, Massachusetts Institute of Technology; kedlaya@mit.edu
Department of Mathematics, University of California, San Diego

Galois representations in arithmetic and geometry
ICM 2010 satellite conference
International Centre, Goa, August 12, 2010

Based on joint work with Ruochuan Liu; manuscript in preparation. Slides available at http://math.mit.edu/~kedlaya/papers/talks.shtml; see there also for slides from my related ICM lecture.

Supported by NSF, DARPA, MIT, IAS.

Contents

Contents

A few period rings

Let K be a complete *unramified* discretely valued field of mixed characteristics $(0, p)$ with valuation subring $\mathfrak{o}_K$ and perfect residue field k. Let G_K denote the absolute Galois group of K.

One describes p-adic Galois representations of G_K using *period rings*.

- $\mathbf{A}_K$: the p-adic completion of $\mathfrak{o}_K((\pi))$.
- $\mathbf{A}_K^\dagger$: the *overconvergent subring* of $\mathbf{A}_K$, consisting of series convergent in some range $* \leq |\pi| < 1$.
- $\mathbf{B}_K = \mathbf{A}_K[1/p]$, $\mathbf{B}_K^\dagger = \mathbf{A}_K^\dagger[1/p]$. These are 2-D local fields.
- $\mathbf{C}_K = \mathbf{B}_{\mathrm{rig},K}^\dagger$: the *Robba ring* of germs of analytic functions on annuli of the form $* \leq |\pi| < 1$. It is a *Bézout ring*, i.e., each finitely generated ideal is principal.

We have inclusions $\mathbf{B}_K^\dagger \subset \mathbf{B}_K, \mathbf{C}_K$, but no comparison of $\mathbf{B}_K$ with $\mathbf{C}_K$.

(φ, Γ)-modules

The period rings carry endomorphisms φ, γ for $\gamma \in \Gamma = \mathbb{Z}_p^\times$:

$$\varphi\left(\sum_i c_i \pi^i\right) = \sum_i \varphi_K(c_i)((1+\pi)^p - 1)^i$$
$$\gamma\left(\sum_i c_i \pi^i\right) = \sum_i c_i((1+\pi)^\gamma - 1)^i.$$

A (φ, Γ)-*module* is a finite free module M equipped with commuting semilinear actions of φ and Γ, with the φ-action induced by an isomorphism $\varphi^* M \cong M$, and the Γ-action continuous for an appropriate topology (weak topology for $\mathbf{B}_K, \mathbf{B}_K^\dagger$; limit-of-Fréchet topology for $\mathbf{B}_K^\dagger, \mathbf{C}_K$). Such an object is *étale* if it admits a basis on which φ acts via a matrix over $\mathbf{A}_K$ (for $\mathbf{B}_K$) or $\mathbf{A}_K^\dagger$ (for $\mathbf{B}_K^\dagger, \mathbf{C}_K$).

Galois representations and (φ, Γ)-modules

Theorem (Fontaine, Cherbonnier-Colmez, Berger, K, etc.)

The following categories are equivalent.

(i) *The category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces.*

(ii) *The category of étale (φ, Γ)-modules over $\mathbf{B}_K$.*

(iii) *The category of étale (φ, Γ)-modules over $\mathbf{B}_K^\dagger$.*

(iv) *The category of étale (φ, Γ)-modules over $\mathbf{C}_K$.*

The same also holds for K replaced by any finite extension, using the unramified case to define analogues of $\mathbf{B}_K, \mathbf{B}_K^\dagger, \mathbf{C}_K$. From (φ, Γ)-modules, one recovers important invariants (e.g., Weil-Deligne representations).

The construction uses the cyclotomic tower; one can use other Lubin-Tate towers instead (or even non-Galois towers, as in Kisin) but we won't do so.

What this talk is not: arithmetic relative theory

One can try to study representations of G_K on finite locally free modules over K-affinoid algebras. There are again fully faithful functors from these representations to étale (φ, Γ)-modules (Berger-Colmez), which are essentially surjective locally (Dee, K-Liu) but not globally (Chenevier). These appear around p-adic local Langlands (Breuil, Colmez, Emerton, etc.) and eigenvarieties (Bellaïche-Chenevier, Liu, Pottharst, etc.).

There are some serious gaps in the theory of slopes for φ-modules over the relative Robba rings arising in this theory. For instance, we do not know whether a pointwise étale φ-module is globally étale, even if the pointwise condition is formulated using Berkovich points. This is complicated by the fact that the étale condition is not open, although it is open around rigid analytic points (K-Liu).

We make no further comments on this subject in this lecture. See also Pappas's lecture later this week.

What this talk is: geometric relative theory

Let A be a K-affinoid algebra in the sense of Berkovich, i.e., a quotient of some generalized Tate algebra $K\{T_1/p_1, \dots, T_n/p_n\}$ (the power series convergent on the closed polydisc of radii $p_1, \dots, p_n$). Let $\mathcal{M}(A)$ denote the associated Berkovich analytic space, i.e., the (compact) space of bounded multiplicative seminorms on A for the product topology. Each point $\alpha \in \mathcal{M}(A)$ has a *residue field* $\mathcal{H}(\alpha)$, the completion of $\mathrm{Frac}(A/\ker(\alpha))$.

On $\mathrm{Spec}(A)$, consider *étale* $\mathbb{Q}_p$*-local systems*, i.e., sheaves in finite dimensional $\mathbb{Q}_p$-vector spaces for the étale topology which locally admit $\mathbb{Z}_p$-lattices which are lisse sheaves (sheaves which are locally constant modulo every power of p). One gets the same category working on $\mathcal{M}(A)$.

When A is connected, étale local systems may be identified with continuous representations of the étale fundamental group of $\mathrm{Spec}(A)$ or $\mathcal{M}(A)$ on finite dimensional $\mathbb{Q}_p$-vector spaces. In particular, for $A = K$, this is what we considered before.

A generalized (partial) (φ, Γ)-module theorem

Fix a *framed K-affinoid algebra* (A, ψ), meaning that $\psi : K\{J\} \to A$ is a map of Banach algebras for some finite set J which defines $\mathcal{M}(A)$ as a closed analytic subspace of a rational subspace of the unit polydisc.

We construct period rings $\mathbf{B}_\psi, \mathbf{B}_\psi^\dagger, \mathbf{C}_\psi$ carrying actions of φ and the group $\Gamma_J = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^J$, define étale (φ, Γ_J)-modules over these rings, and prove the following.

Theorem

Consider the following categories.

(i) *The category of étale $\mathbb{Q}_p$-local systems on* $\mathrm{Spec}(A)$.

(ii) *The category of étale (φ, Γ_J)-modules over $\mathbf{B}_\psi$.*

(iii) *The category of étale (φ, Γ_J)-modules over $\mathbf{B}_\psi^\dagger$.*

(iv) *The category of étale (φ, Γ_J)-modules over $\mathbf{C}_\psi$.*

Then (i)$\Leftrightarrow$(ii), (iii)$\Leftrightarrow$(iv), and (iii)$\Rightarrow$(ii) is fully faithful.

Functoriality

The previous theorem includes functoriality with respect to diagrams

$$\begin{array}{ccc} K\{J\} & \xrightarrow{\psi} & A \\ \downarrow & & \downarrow \\ K\{J'\} & \xrightarrow{\psi'} & B \end{array}$$

in which $A \to B$ is a homomorphism of K-affinoid algebras and $K\{J\} \to K\{J'\}$ is induced by a map $J \to J'$ (i.e., a morphism of affine toric varieties). For $H = \ker(\Gamma_{J'} \to \Gamma_J)$, we have maps

$$\mathbf{B}_\psi \to \mathbf{B}^H_{\psi'}, \qquad \mathbf{B}^\dagger_\psi \to (\mathbf{B}^\dagger_{\psi'})^H, \qquad \mathbf{C}_\psi \to \mathbf{C}^H_{\psi'}.$$

Finite étale algebras

Theorem

In the previous diagram, suppose $J \subseteq J'$ and B is finite étale over A. Consider the following categories.

(i) *The category of étale $\mathbb{Q}_p$-local systems on* $\mathrm{Spec}(B)$.

(ii) *The category of étale (φ, Γ_J)-modules over $\mathbf{B}^H_{\psi'}$.*

(iii) *The category of étale (φ, Γ_J)-modules over $(\mathbf{B}^\dagger_{\psi'})^H$.*

(iv) *The category of étale (φ, Γ_J)-modules over $\mathbf{C}^H_{\psi'}$.*

Then (i)$\Leftrightarrow$(ii), (iii)$\Leftrightarrow$(iv), and (iii)$\Rightarrow$(ii) is fully faithful.

If $A = K$, then $\mathbf{B}_K = \mathbf{B}^H_{\psi'}$ and so on, and we recover the usual theory of (φ, Γ)-modules *except* for Cherbonnier-Colmez. (ii)$\Rightarrow$(iii) probably requires Sen decompletion (as in Berger-Colmez) plus ramification theory for local fields with imperfect residue field (Abbes-Saito). But (i)$\Rightarrow$(ii) doesn't!

Application: Local systems on Rapoport-Zink period spaces

One can construct interesting local systems over analytic spaces interpolating between Galois representations. E.g., fix a finite-dimensional K-vector space V equipped with an isomorphism $\varphi^* V \cong V$ (an *isocrystal* or *φ-module*). For L a finite extension of K, an exhaustive decreasing filtration on $V \otimes_K L$ defines a crystalline Galois representation if and only if it is *weakly admissible* (Colmez-Fontaine, Berger, Kisin).

The moduli space of exhaustive decreasing filtrations with fixed jumps (or *Hodge-Tate weights*) is a symmetric space, of which the weakly admissible locus is an open Berkovich subspace. On a smaller open with the same rigid analytic points (the *admissible locus*), we form an étale $\mathbb{Q}_p$-local system interpolating the crystalline Galois representations from above. This uses a relative form of Berger's construction (inspired by work of Hartl): form a certain nonétale (φ, Γ_J)-module and cut out the *étale locus*.

More on this in my ICM lecture.

Contents

Overview

Our constructions are built using Witt vector rings for perfect $\mathbb{F}_p$-algebras. These largely replace the Fontaine-Wintenberger fields of norms constructions used in the original development of p-adic Hodge theory.

This is in the same spirit as the Fargues-Fontaine development of p-adic Hodge theory using coherent sheaves on a one-dimensional Berkovich analytic space, as described in the previous lecture.

Witt vectors

Let R be a perfect $\mathbb{F}_p$-algebra, i.e., a ring in which $p = 0$ and the Frobenius map $\overline{\varphi} : x \mapsto x^p$ is bijective. There is then a unique ring $W(R)$ which is p-adically completed and separated and carries an isomorphism $W(R)/(p) \cong R$.

One can describe $W(R)$ using the *Teichmüller map*, a multiplicative map $[\cdot] : R \to W(R)$, characterized by the property that $[\overline{x}]$ has a p^n-th root for $n = 0, 1, \dots$. Each $W(R)$ admits a unique representation

$$\sum_{n=0}^{\infty} p^n [\overline{x}_n] \qquad (\overline{x}_n \in R)$$

and one can do arithmetic in terms of these representations using certain universal polynomials constructed by Witt.

Raising and lowering norms

Any (power-)multiplicative norm α on R bounded by the trivial norm lifts to a (power-)multiplicative norm $\lambda(\alpha)$ on $W(R)$:

$$\lambda(\alpha)\left(\sum_{n=0}^{\infty} p^n[\overline{x}_n]\right) = \max\{p^{-n}\alpha(\overline{x}_n)\}.$$

Any (power-)multiplicative norm β on $W(R)$ bounded by the p-adic norm projects to a (power-)multiplicative norm $\mu(\beta)$ on R:

$$\mu(\beta)(\overline{x}) = \beta([\overline{x}]).$$

These define continuous maps between the analytic spaces $\mathcal{M}(R)$ (for the trivial norm) and $\mathcal{M}(W(R))$ (for the p-adic norm). In fact, the two spaces are homotopy equivalent.

Relative extended Robba rings

Say R carries a power-multiplicative norm α. Let $\mathfrak{o}_R$ be the subring on which α is bounded by 1.

For $r > 0$, the set of $\sum_{n=0}^{\infty} p^n[\overline{x}_n]$ for which $\lambda(\alpha^r)(p^{-n}[\overline{x}_n]) \to 0$ forms a ring $\tilde{\mathcal{R}}_R^{\mathrm{inte},r}$ to which $\lambda(\alpha^r)$ (originally defined only on $W(\mathfrak{o}_R)$) extends. Put $\tilde{\mathcal{R}}_R^{\mathrm{inte}} = \cup_{r>0} \tilde{\mathcal{R}}_R^{\mathrm{inte},r}$.

Let $\tilde{\mathcal{R}}_R^r$ be the Fréchet completion of $\tilde{\mathcal{R}}_R^{\mathrm{inte},r}[1/p]$ for $\lambda(\alpha^s)$ for $s \in (0, r]$. Put $\tilde{\mathcal{R}}_R = \cup_{r>0} \tilde{\mathcal{R}}_R^r$; we call this a *relative extended Robba ring*. It carries an extension of the Witt vector Frobenius φ (induced by $\overline{\varphi}$ on R).

A *φ-module* over $\tilde{\mathcal{E}}_R = W(R)[1/p]$, $\tilde{\mathcal{R}}_R^{\mathrm{bd}} = \tilde{\mathcal{R}}_R^{\mathrm{inte}}[1/p]$, or $\tilde{\mathcal{R}}_R$ is a finite locally free module M equipped with an isomorphism $\varphi^* M \cong M$. (This is slightly inaccurate: φ-modules over $\tilde{\mathcal{R}}_R$ should really be defined as "vector bundles", but we'll gloss over that subtlety throughout this talk.)

Slope theory over extended Robba rings

Let M be a φ-module over $\tilde{\mathcal{R}}_R$. For R a field, define $\deg(M)$ as the p-adic valuation of the matrix on which φ acts on some (any) basis of M, and the *slope* $\mu(M) = \deg(M)/\operatorname{rank}(M)$ (when $M \neq 0$). M is *étale* if on some basis, φ acts via an invertible matrix over $\tilde{\mathcal{R}}_R^{\mathrm{inte}}$.

Theorem (K, 2005; also Fargues-Fontaine, 2010)

M is étale iff $\mu(M) = 0$ and M has no φ-submodule of negative slope.

For general R, the *étale locus* of M is the set of $\beta \in \mathcal{M}(R)$ for which $M \otimes_{\tilde{\mathcal{R}}_R} \tilde{\mathcal{R}}_{\mathcal{H}(\beta)}$ is étale.

Theorem

If β is in the étale locus, then M admits an étale basis over some open neighborhood of β. Consequently, the étale locus is open.

This fails in the arithmetic relative theory (Robba rings over an affinoid)!

The theta map

For A a p-adically separated and complete ring, the *inverse perfection* A^{frep} of A is the inverse limit of $A/(p)$ under Frobenius. That is, it consists of sequences $(\dots, \overline{x}_1, \overline{x}_0)$ in $A/(p)$ with $\overline{x}_{i+1}^p = \overline{x}_i$.

The *theta map* $\theta : W(A^{\mathrm{frep}}) \to A$ is a homomorphism sending $\sum_{i=0}^{\infty} p^i [\overline{x}_i]$ to $\lim_{j \to \infty} p^i x_{ij}^{p^j}$, where $(\dots, x_{i1}, x_{i0})$ is any sequence lifting $\overline{x}_i$. Any power-multiplicative norm β on A bounded by the p-adic norm induces a norm $\alpha = \mu(\theta^*(\beta))$ on A^{frep}.

For $\epsilon_0, \epsilon_1, \dots$ a sequence in A with $\epsilon_0 = 1$, $\epsilon_1 \neq 1$, and $\epsilon_{n+1}^p = \epsilon_n$, put $\overline{\pi} = (\dots, \epsilon_1 - 1, \epsilon_0 - 1) \in A^{\mathrm{frep}}$. Then $\ker(\theta)$ is generated by

$$z = \sum_{i=0}^{p-1} [\overline{\pi} + 1]^{i/p}.$$

The theta map and étale extensions

Let $\mathbf{FEt}(*)$ denote the category of finite étale algebras over a ring $*$. Keep notation from the previous slide, and put $R = A^{\mathrm{frep}}[1/\overline{\pi}]$. Then $\mathbf{FEt}(R)$ and $\mathbf{FEt}(\tilde{\mathcal{R}}_R^{\mathrm{inte}})$ are equivalent because $\tilde{\mathcal{R}}_R^{\mathrm{inte}}$ is Henselian. They are also equivalent to $\mathbf{FEt}(\tilde{\mathcal{R}}_R^{\mathrm{inte},r})$ for any $r > 0$.

Theorem

If R is the completed perfection of an affinoid algebra over a complete field, then $\theta : \tilde{\mathcal{R}}_R^{\mathrm{inte},1} \to A$ induces an equivalence $\mathbf{FEt}(R) \sim \mathbf{FEt}(A[1/p])$.

The hard part is essential surjectivity. If R is a field, we may reduce to the case of an unramified, tamely ramified, or wildly ramified $(\mathbb{Z}/p\mathbb{Z})$-extension. In general, transfer étale algebras locally and invoke Kiehl's theorem.

This construction is key to the passage from étale $\mathbb{Q}_p$-local systems to (φ, Γ_J)-modules over $\mathbf{B}_\psi$, replacing fields of norms.

Contents

Setup

For J a finite set, equip $\mathfrak{o}_K\{J\}[\![\pi]\!]$ with

- the Gauss norm;
- a Frobenius lift φ acting as φ_K on K, taking each $T \in J$ to T^p, and taking π to $(1+\pi)^p - 1$;
- a K-linear action of $\Gamma_J = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^J$ with $\gamma \in \mathbb{Z}_p^\times$ taking π to $(1+\pi)^\gamma - 1$ (fixing J) and $1 \in \mathbb{Z}_p^T$ taking T to $(1+\pi)T$ (fixing π and the rest of J).

These also act on $\mathbb{D}_J$, the open unit π-disc over $\mathcal{M}(K\{J\})$.

Perfect period rings

Fix a framed K-affinoid algebra (A, ψ). Let $A_{\psi,n}$ be the reduced quotient of

$$A \otimes_{K\{J\}} K\{J\}$$

for the finite map $K\{J\} \to K\{J\}$ acting as φ_K^n on K and taking $T \in J$ to T^{p^n}. Equip $A_{\psi,n}$ with the power-multiplicative *spectral norm*. There are natural isometries $A_{\psi,n} \to A_{\psi,n+1}$; let $A_{\psi,\infty}$ be the completed direct limit of the $A_{\psi,n}$. Form $\overline{A}_\psi$ by taking the inverse perfection of the valuation subring of $A_{\psi,\infty}$, then inverting $\overline{\pi}$. For $R = \overline{A}_\psi$, we now set

$$\tilde{\mathbf{A}}_\psi = W(R),\ \tilde{\mathbf{A}}_\psi^\dagger = \tilde{\mathcal{R}}_R^{\mathrm{inte}},\ \tilde{\mathbf{B}}_\psi = \tilde{\mathcal{E}}_R,\ \tilde{\mathbf{B}}_\psi^\dagger = \tilde{\mathcal{R}}_R^{\mathrm{bd}},\ \tilde{\mathbf{C}}_\psi = \tilde{\mathcal{R}}_R.$$

Imperfect period rings

Let $\mathbf{C}_\psi$ be the completed localization of $\mathfrak{o}_K\{J\}[\![\pi]\!]$ in $\tilde{\mathbf{C}}_\psi$ for the limit-of-Fréchet topology. Put $\mathbf{A}^\dagger_\psi = \tilde{\mathbf{A}}^\dagger_\psi \cap \mathbf{C}_\psi$, $\mathbf{B}^\dagger_\psi = \tilde{\mathbf{B}}^\dagger_\psi \cap \mathbf{C}_\psi$. Define $\mathbf{A}_\psi, \mathbf{B}_\psi$ by p-adic completion of $\mathbf{A}^\dagger_\psi, \mathbf{B}^\dagger_\psi$.

Lemma

For $0 < s \leq r$, let $\mathbf{C}^{[s,r]}_\psi$ be the completion of $\mathbf{C}^r_\psi = \mathbf{C}_\psi \cap \tilde{\mathbf{C}}^r_\psi$ for the norm $\max\{\lambda(\alpha^r), \lambda(\alpha^s)\}$. Then for r sufficiently small, $\mathbf{C}^{[s,r]}_\psi$ is a K-affinoid algebra. Moreover, $\mathbf{A}_\psi/(p)$ is a $k((\overline{\pi}))$-affinoid algebra.

One first proves this when $\mathcal{M}(A)$ is a rational subspace of $\mathcal{M}(K\{J\})$. For general (A, ψ), by construction, A receives a surjection from a K-affinoid algebra B for which $\mathcal{M}(B)$ is a rational subspace of $\mathcal{M}(K\{J\})$. This induces surjections from period rings for B to period rings for A.

(φ, Γ_J)-modules

The period rings inherit actions of φ and Γ_J. We use these to define (φ, Γ_J)-*modules* over $\mathbf{B}_\psi, \mathbf{B}^\dagger_\psi, \mathbf{C}_\psi$. (Again, for $\mathbf{C}_\psi$, one considers not modules but vector bundles.)

One catch: the action of the factor of $\mathbb{Z}_p^J$ corresponding to $T \in J$ is required to act trivially modulo T. This restriction is needed to construct descent data, because the map $K\{J\} \to K\{J\}$ taking T to T^{p^n} is not étale over the zero locus of T.

The étale condition can be imposed in several (nontrivially) equivalent ways: pointwise, locally, or after passing to the corresponding perfect period ring.

Proof of the main theorem

The equivalence between étale $\mathbb{Q}_p$-local systems and étale (φ, Γ_J)-modules over $\mathbf{B}_\psi$ proceeds as in the original case, using the equivalence $\mathbf{FEt}(A_{\psi,\infty}) \sim \mathbf{FEt}(\mathbf{A}_\psi/(p))$, Katz's description of unit-root F-crystals in terms of étale local systems in positive characteristic, and faithfully flat descent.

The equivalence between $\mathbf{B}^\dagger_\psi$ and $\mathbf{C}_\psi$, and the full faithfulness from $\mathbf{B}^\dagger_\psi$ to $\mathbf{B}_\psi$, uses Katz's description plus some elementary calculations (as in the original case).

Decompletion?

The big open problem is the passage from $\mathbf{B}_\psi$ to $\mathbf{B}_\psi^\dagger$. This should require Sen-Tate decompletion and some ramification theory. In case $\mathcal{M}(A)$ is smooth, it may be possible to reduce to the theorem of Andreatta and Brinon, which handles (among others) the case $A = K\{J^{\pm 1}\}$ using decompletion and Faltings's almost purity theorem.

It may also be possible to reduce to the smooth case by arguing that any étale $\mathbb{Q}_p$-local system over a closed analytic subspace of an affinoid space spreads over a neighborhood. Such spreading happens for finite étale covers, and for crystalline or log-crystalline local systems because the admissible locus is open in the period domain.

Other invariants?

It should be possible to recover other invariants of a local system from a corresponding (φ, Γ_J)-module. For instance, it should be possible to compute étale cohomology by analogy with the corresponding construction for a representation (Herr, Liu).