Convergence of solutions of p-adic differential equations

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Slides: `http://math.mit.edu/~kedlaya/papers/talks.shtml`.
Work in progress, but see my book *p-adic Differential Equations* (2010).

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2. A bit of Berkovich spaces

3. Radii of convergence

Motivation: complex differential equations

Let $f_0, \dots, f_{n-1}$ be holomorphic functions on some simply connected domain $U \subseteq \mathbb{C}$. Then the differential equation

$$y^{(n)} + f_{n-1} y^{(n-1)} + \cdots + f_0 y = 0$$

has a full set of solutions in the space of holomorphic functions on U. In particular, for $z_0 \in U$, the formal power series solutions at z_0 converge on the largest disc centered at z_0 contained in U.

Motivation: p-adic differential equations

By contrast, consider the differential equation $y' - y = 0$, but consider p-adic power series solutions at 0. These solutions are multiples of the exponential function, which has radius of convergence $p^{-1/(p-1)} < \infty$ despite the absence of any singularities!

By contrast, there exist many examples of differential equations whose p-adic power series solutions do converge as far as can reasonably be expected. For instance, this is typical for Picard-Fuchs equations (also known as Gauss-Manin connections).

Dwork et al. studied this phenomenon by considering "convergence in generic discs". In modern language, this can be described in terms of the geometry of some Berkovich analytic spaces. This can be done for general Berkovich curves, but for this talk we discuss only the case of discs.

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Berkovich discs

Let K be a field complete for a nonarchimedean absolute value, e.g., $K = \mathbb{Q}_p$, or $K = \mathbb{C}_p$ is a completed algebraic closure of $\mathbb{Q}_p$.

For $r > 0$, the *r-Gauss norm* $|\cdot|_r$ on $K[T]$ is defined by

$$|a_0 + a_1 T + \cdots + a_n T^n|_r = \max_i \{|a_i| r^i\}.$$

Complete to get $K\{T/r\} \subseteq K[\![T]\!]$ (a *Tate algebra*).

The *Berkovich closed disc* $\mathbb{D}_{r,K}$ of radius r is the set of multiplicative seminorms on $K\{T/r\}$ which are bounded by $|\cdot|_r$. This set is topologized as a subset of the product $\mathbb{R}^{K\{T/r\}}$; it is thus compact.

The geometry of this space turns out to be much simpler than the previous definition might suggest! We will see a picture later.

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Some points inside a Berkovich disc

For $a \in K$ with $|a| \leq r$ and $\rho \in [0, r]$, let $\zeta_{a,\rho} \in \mathbb{D}_{r,K}$ be the ρ-Gauss norm in the parameter $T - a$. This acts as a generic point of the disc $|z - a| \leq \rho$. (For $\rho = 0$, this is the evaluation-at-a seminorm.)

Not every point of $\mathbb{D}_{r,K}$ is of this form (more on this later). However, every point of $\mathbb{D}_{r,K}$ is the restriction of a point of this form over some larger field, by the following construction.

For $\alpha \in \mathbb{D}_{r,K}$, let $\mathcal{H}(\alpha)$ be the α-completion of $\mathrm{Frac}(K\{T/r\}/\ker(\alpha))$. Let $x_\alpha \in \mathcal{H}(\alpha)$ be the image of T. Then α is the restriction of $\zeta_{x_\alpha,0} \in \mathbb{D}_{r,\mathcal{H}(\alpha)}$. (One might view α as a generic point of a disc centered at x_α.)

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Some segments inside a Berkovich disc

For any $a \in K$, the map $\rho \mapsto \zeta_{a,\rho}$ defines a continuous injection $[0, r] \to \mathbb{D}_{r,K}$. Using extension of scalars, for any $\alpha \in \mathbb{D}_{r,K}$ we can construct a segment from α to the *Gauss point* $\zeta_{0,r}$. A *skeleton* in $\mathbb{D}_{r,K}$ is a finite union of such segments; it is homeomorphic to a tree.

Theorem

The space $\mathbb{D}_{r,K}$ is homeomorphic to the inverse limit of its skeleta. In particular, it is contractible.

A picture for $r = 1$

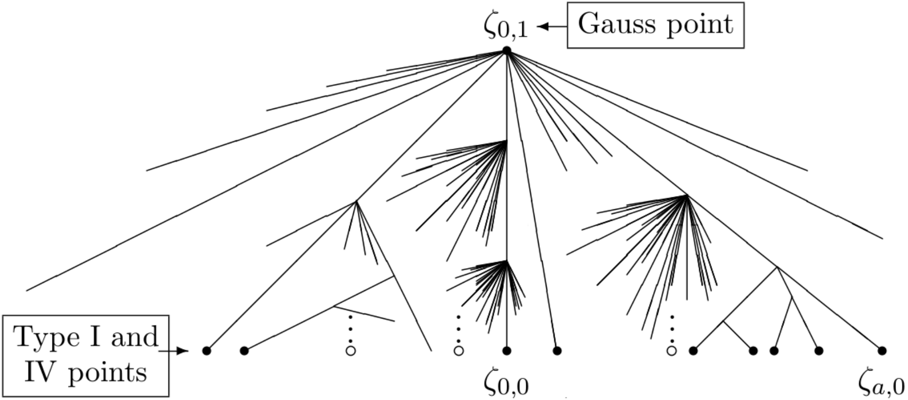

Illustration stolen from Joe Silverman via Matt Baker.

Points of Berkovich discs

When K is algebraically closed, points of a Berkovich closed disc are traditionally classified as follows.

- I A seminorm $\zeta_{a,0}$ for some $a \in K$.
- II A norm $\zeta_{a,\rho}$ for some $a \in K$ and $\rho \in |K^\times|$.
- III A norm $\zeta_{a,\rho}$ for some $a \in K$ and $\rho \notin |K^\times|$.
- IV A point not equal to any $\zeta_{a,\rho}$. Such points act as generic points of "virtual discs" which contais no points of K; these cannot exist if K is spherically complete (which $\mathbb{C}_p$ is not!).

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Differential modules

Instead of a differential equation, it is more convenient to use a more algebraic structure. A first step is to consider a system of first-order linear differential equations:

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}' = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

A second step is to consider a finite free module M over a ring R such that R carries a derivation d and M carries an additive map D such that

$$D(rm) = d(r)m + rD(m).$$

Such a structure will be called a *differential module* over R. Finding solutions of a differential equation then corresponds to finding elements in the kernel of D, also called *horizontal sections*.

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Local horizontal sections

Assume from now on that K is of characteristic 0. Let M be a nonzero differential module over $K\{T/r\}$. For each $\alpha \in \mathbb{D}_{r,K}$, let $M_{\alpha,0}$ be the extension of scalars of M along the homomorphism

$$K\{T/r\} \to \mathcal{H}(\alpha)[\![T - x_\alpha]\!].$$

Then $M_{\alpha,0}$ admits a basis in the kernel of D; that is, the natural map

$$M_{\alpha,0}^{D=0} \otimes_{\mathcal{H}(\alpha)} \mathcal{H}(\alpha)[\![T - x_\alpha]\!] \to M_{\alpha,0}$$

is an isomorphism. The elements of $M_{\alpha,0}^{D=0}$ may be viewed as *formal horizontal sections* of M at α. (If we started with a differential equation, these corerspond to formal power series solutions at a point.)

Radii of convergence

For $\rho \in (0, r]$, let $M_{\alpha,\rho}$ be the extension of scalars of M along the homomorphism

$$K\{T/r\} \to \mathcal{H}(\alpha)\{(T - x_\alpha)/\rho\}.$$

For ρ sufficiently small, we have $M_{\alpha,\rho}^{D=0} = M_{\alpha,0}^{D=0}$, i.e., the formal horizontal sections at α converge in some disc. (This is already nontrivial!)

Put $n = \operatorname{rank}(M)$. For $i = 1, \ldots, n$, let $s_i(M, \alpha)$ be the supremum of those ρ for which $\dim_K(M_{\alpha,\rho}^{D=0}) \geq n - i + 1$, i.e., the radius of the maximal open disc around x_α on which M admits $n - i + 1$ linearly independent horizontal sections. We have

$$0 < s_1(M, \alpha) \leq \cdots \leq s_n(M, \alpha).$$

Handy fact: these radii are invariant under enlarging K.

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Continuity of the radii of convergence

All of the following results are due to Baldassarri for $i = 1$ and are new for $i > 1$. However, the techniques build upon 50+ years of prior work (by Dwork, Robba, Christol, Young, Pons, etc.)

Theorem (Baldassarri for $i = 1$, K for $i > 1$)

For $i = 1, \dots, n$, the function $s_i(M, \alpha)$ on $\mathbb{D}_{r,K}$ is continuous. Moreover, there exists a skeleton S in $\mathbb{D}_{r,K}$ such that for every $\alpha \in \mathbb{D}_{r,K}$, $s_i(M, \alpha) = s_i(M, \beta)$ for β the first point at which the path from α to the Gauss point meets S.

The minimal such S is sometimes called the *controlling polygon* of M.

Integrality of the radii of convergence

From now on, let p be the characteristic of the residue field of K.

Theorem (Baldassarri for $i = 1$, K for $i > 1$)

Suppose either that $p = 0$ or that $p > 0$ and $|K^\times|$ is p-divisible. Then for all $a \in K$, $\rho \in (0, r]$, and $i \in \{1, \dots, n\}$,

$$s_i(M, \zeta_{a,\rho})^{n!} \in |K^\times| \cdot \rho^{\mathbb{Z}}.$$

Moreover, if $i = n$ or if $i < n$ and $s_i(M, \zeta_{a,\rho}) < \min\{\rho, s_{i+1}(M, \zeta_{a,\rho})\}$, then

$$\prod_{j=1}^{i} s_j(M, \zeta_{a,\rho}) \in |K^\times| \cdot \rho^{\mathbb{Z}}.$$

Subharmonicity of the radii of convergence

Theorem (Baldassarri for $i = 1$, K for $i > 1$)

For any $i \in \{1, \ldots, n\}$ and any $a \in K$, the function $\rho \mapsto s_i(M, \zeta_{a,\rho})$ is nondecreasing and log-concave.

Theorem (Baldassarri for $i = 1$, K for $i > 1$)

For any $i \in \{1, \ldots, n\}$, the function $\rho \mapsto -\log s_i(M, \zeta_{a,\rho})$ is **subharmonic***: at any vertex of the skeleton S, the slope “from above” (which is ≤ 0) is at most the sum of the slopes “from below”.*

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Some proof techniques

- By extension of scalars, we can always reduce to looking at $\zeta_{a,\rho}$.
- If $s_i(M, \zeta_{a,\rho}) > \rho$, then $s_i(M, \zeta_{a,\rho}) = s_i(M, \zeta_{a,\rho+\epsilon})$ since the discs in question coincide.
- Put $\omega = 1$ if $p = 0$ or $\omega = p^{-1/(p-1)}$ if $p > 0$. If $s_i(M, \zeta_{a,\rho}) < \omega\rho$, then $s_i(M, \zeta_{a,\rho})$ can be read off from a certain Newton polygon.
- If $\omega\rho \leq s_i(M, \zeta_{a,\rho}) < \rho$, then we can study $s_i(M, \zeta_{a,\rho})$ by pushing forward along the map $T - a \mapsto (T - a)^p$; this has the effect of moving $s_i(M, \zeta_{a,\rho})$ closer to the visible range.
- If $s_i(M, \zeta_{a,\rho}) = \rho$, it is hard to get any information directly; we must work around this!

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Some stronger results

Theorem (Baldassarri for $i = 1$, K for $i > 1$)

Suppose either $p = 0$ or that $p > 0$, $|K^\times|$ is p-divisible, and (see below).

(a) *The functions $s_i(M, \alpha)$ are constant near any point of type IV.*

(b) *If $i = n$ or if $i < n$ and $s_i(M, \zeta_{a,\rho}) < s_{i+1}(M, \zeta_{a,\rho})$, then*

$$\prod_{j=1}^{i} s_j(M, \zeta_{a,\rho}) \in |K^\times| \cdot \rho^{\mathbb{Z}}.$$

(c) *The functions $s_i(M, \alpha)^{n!}$ and $\prod_{i=1}^{n} s_i(M, \alpha)$ each have the form $\alpha \mapsto \min\{\alpha(f_1), \ldots, \alpha(f_m)\}$ for some $f_1, \ldots, f_m \in K\{T/r\}$.*

If $p > 0$, one must assume that $\mathrm{rank}(M) < p$ or that "no exponent differences are p-adic Liouville numbers." For example, this is true for Picard-Fuchs equations (which have *Frobenius structures*).

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$$\prod_{j=1}^{i} s_j(M, \zeta_{a,\rho}) \in |K^\times| \cdot \rho^{\mathbb{Z}}.$$

(c) *The functions $s_i(M, \alpha)^{n!}$ and $\prod_{i=1}^{n} s_i(M, \alpha)$ each have the form $\alpha \mapsto \min\{\alpha(f_1), \dots, \alpha(f_m)\}$ for some $f_1, \dots, f_m \in K\{T/r\}$.*

If $p > 0$, one must assume that $\operatorname{rank}(M) < p$ or that "no exponent differences are p-adic Liouville numbers." For example, this is true for Picard-Fuchs equations (which have *Frobenius structures*).