The p-adic arithmetic curve: algebraic and analytic aspects

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Context of this talk: the *hypothetical* arithmetic curve

The properties of zeta functions and L-functions of algebraic varieties over finite fields (e.g., Weil's conjectures) are well explained by *cohomology theories* (étale cohomology, rigid p-adic cohomology). These provide *spectral interpretations* of zeros and poles as eigenvalues of Frobenius on certain vector spaces.

It is suspected that properties of zeta functions and L-functions over $\mathbb{Z}$ can be similarly explained by describing an *arithmetic curve* and (foliated) cohomology thereof. Rather than a discrete Frobenius operator, one should instead find a one-parameter flow (time evolution) with a simple periodic orbit of length $\log p$ contributing an Euler factor at p.

A few formal properties of this picture are realized by the *Bost-Connes system*, in which Riemann ζ appears as a quantum-statistical partition function.

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A p-adic arithmetic curve

In this talk, we describe results from *p-adic Hodge theory* which provide a *curve* resembling the periodic orbit corresponding to p in a putative arithmetic curve. There is also some formal resemblance to the p-adic BC system.

This *p-adic arithmetic curve* admits coefficient objects corresponding to motives over $\mathbb{Q}_p$, from which étale and de Rham cohomology can be read off naturally. (These are closely related to (φ, Γ)-*modules*.) This suggests the possibility of building an arithmetic curve *with coefficients* so as to provide a spectral interpretation of global zeta and L-functions.

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What is Hodge theory?

An algebraic variety over $\mathbb{C}$ admits both Betti (singular) and algebraic de Rham cohomologies, which are related by a *comparison isomorphism*. This provides the same $\mathbb{C}$-vector space H^i with both a $\mathbb{Z}$-lattice and a Hodge filtration. For example, if E is an elliptic curve, then H^1 has dimension 2. The $\mathbb{Z}$-structure on H^1 projects to a lattice in the 1-dimensional space $\mathrm{Fil}^0 / \mathrm{Fil}^1$, the quotient by which is E.

Ordinary Hodge theory consists (in part) of studying the relationship between integral structures and filtrations, abstracted away from algebraic varieties.

p-adic Hodge theory

Over a finite extension K of $\mathbb{Q}_p$, Fontaine discovered deep relationships between p-adic étale cohomology and algebraic de Rham cohomology. However, in this case, these are related over some surprisingly large *p-adic period rings*.

One important application is to characterize p-adic Galois representations which can arise from étale cohomology (e.g., Fontaine-Mazur conjecture). This characterization is built into most current results on modularity of Galois representations (e.g., Khare-Wintenberger's proof of Serre's conjecture).

One also embeds continuous p-adic representations of G_K into a larger category of (φ, Γ)-*modules* in which irreducible representations may fail to remain irreducible. This is not pathological! It occurs for representations occurring in practice (e.g., those attached to p-adic modular forms) and has strong repercussions in the study of *eigenvarieties*.

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Witt vectors

Fix a prime p and let W denote the functor of *p-typical Witt vectors*. For R a perfect $\mathbb{F}_p$-algebra, $W(R)$ is p-adically separated and complete and $W(R)/(p) \cong R$. Also, $W(R)$ admits a multiplicative *Teichmüller map* $r \to [r]$ whose composition with reduction modulo p is the identity.

One can also define *big Witt vectors* over any ring R, by imposing an exotic ring structure on sequences $(x_1, x_2, \dots)$ in a manner functorial in R so that the *ghost map*

$$(x_n)_{n\in\mathbb{N}} \mapsto (w_n)_{n\in\mathbb{N}}, \qquad w_n = \sum_{d|n} d x_d^{n/d}$$

defines a ring homomorphism to the ordinary product $R^{\mathbb{N}}$. Retaining components indexed by powers of p reproduces the p-typical construction. The big Witt vectors always form a *λ-ring*.

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Some rings in p-adic Hodge theory

Let F^+, F be the completed perfections of $\mathbb{F}_p[[\overline{\pi}]], \mathbb{F}_p((\overline{\pi}))$, equipped with the $\overline{\pi}$-adic norm with normalization $|\overline{\pi}| = p^{-p/(p-1)}$. The Witt ring $W(F^+)$ carries for each $r > 0$ a multiplicative *Gauss norm*

$$\left| \sum_{n=0}^{\infty} p^n [\overline{x}_n] \right|_r = \max_n \{ p^{-n} |\overline{x}_n|^r \}.$$

The Frobenius φ on $W(F^+)$ satisfies

$$|\varphi(x)|_r = |x|_{pr}.$$

Let $\mathbb{B}_+$ denote the Fréchet completion of $W(F^+)[p^{-1}]$ with respect to all of the Gauss norms. The group $\Gamma = \mathbb{Z}_p^\times$ acts via

$$\gamma(1+\overline{\pi}) = (1+\overline{\pi})^\gamma = \sum_{i=0}^{\infty} \binom{\gamma}{i} \overline{\pi}^i.$$

The Fargues-Fontaine curve

Let P denote the graded ring

$$P = \bigoplus_{n=0}^{\infty} P_n, \qquad P_n = \mathbb{B}_+^{\varphi = p^n}.$$

The *Fargues-Fontaine curve* is the scheme $\mathrm{Proj}(P)$.

Theorem (Fargues-Fontaine, after K, Berger)

The scheme $\mathrm{Proj}(P)$ *is noetherian of dimension* 1*, regular, and connected. It is also* **complete***: it admits a homomorphism* $\deg : \mathrm{Div}(\mathrm{Proj}(P)) \to \mathbb{Z}$ *which is surjective, nonnegative on effective divisors, and zero on principal divisors. (For* $f \in P_n$ *nonzero,* $\deg(V(f)) = n$*.)*

Vector bundles and Galois representations

Since deg factors through Pic(Proj(P)), we get a well-defined degree function on line bundles. As usual, define the *degree* of a vector bundle as the degree of its top exterior power, and define the *slope* of a nonzero vector bundle as

$$\mu(V) = \frac{\deg(V)}{\mathrm{rank}(V)}.$$

A vector bundle V' is *semistable* if it admits no nonzero proper subbundle V' with $\mu(V') > \mu(V)$.

Theorem (Fargues-Fontaine, after K, Berger)

The category of continuous representations of $G_{\mathbb{Q}_p}$ on finite-dimensional $\mathbb{Q}_p$-vector spaces is equivalent to the category of Γ-equivariant semistable vector bundles of slope 0 on $\mathrm{Proj}(P)$.

(Aside: the interaction between φ and Γ resembles the BC-system.)

Vector bundles and comparison isomorphisms

Suppose that V is the vector bundle corresponding to $H^i_{\mathrm{et}}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ for some smooth proper variety X over $\mathbb{Q}_p$.

One recovers the étale cohomology $H^i_{\mathrm{et}}(X \times_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}, \mathbb{Q}_p)$ by taking Γ-fixed global sections of V.

One recovers the de Rham cohomology $H^i_{\mathrm{dR}}(X, \mathbb{Q}_p)$ by taking Γ-fixed sections of V over the fraction field of the completed local ring of $\mathrm{Proj}(P)$ at the *de Rham point*, the unique vanishing point of

$$t = \log([1+\overline{\pi}]) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i}([1+\overline{\pi}] - 1)^i \in P_1.$$

This point has residue field is the completion of $\mathbb{Q}_p(\mu_{p^\infty})$. Note: every finite étale algebra over $\mathbb{Q}_p(\mu_{p^\infty})$ with Γ-action lifts uniquely to a finite étale cover of $\mathrm{Proj}(P)$ with Γ-action.

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Approaches to nonarchimedean analytic geometry

Analytic spaces over a nonarchimedean field are somehow glued together from Banach algebras (always commutative here, and typically assumed to be *affinoid*). Classically this is done by taking maximal ideals and imposing a Grothendieck topology (Tate's *rigid analytic spaces*).

For this talk, it is better to follow Berkovich and take Gel'fand spectra (spaces of bounded multiplicative real-valued seminorms). These have less disconnected topology; for instance, the "closed unit disc" in this setting is *contractible*. (Related fact: the analytification of a complete curve has homotopy type related to its *semistable reduction*.)

It is better in the long run to add valuations of height greater than 1 (to get *adic spaces* as in Huber or Fujiwara-Kato), but we won't do that today.

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Witt vectors

Let R be a perfect $\mathbb{F}_p$-algebra, equipped with the trivial norm. Equip $W(R)$ with the trivial norm. There are natural maps on Gel'fand spectra:

$$\lambda : \mathcal{M}(R) \to \mathcal{M}(W(R)), \qquad \lambda(\alpha)\left(\sum_{n=0}^{\infty} p^n [\overline{x}_n]\right) = \max_n \{p^{-n}\alpha(\overline{x}_n)\}$$

$$\mu : \mathcal{M}(W(R)) \to \mathcal{M}(R), \qquad \mu(\beta)(\overline{x}_n) = \beta([\overline{x}_n]).$$

Theorem (K)

The maps λ, μ are continuous and preserve rational subspaces. Moreover, there is a natural (in R) vertical (for μ) homotopy on $\mathcal{M}(W(R))$ between id *and $\lambda \circ \mu$.*

That is, $\mathcal{M}(W(R))$ behaves like a disc bundle over $\mathcal{M}(R)$.

Relative circles

Now let R be a perfect *uniform* Banach $\mathbb{F}_p$-algebra with norm α. (Uniformity means $\alpha(x^2) = \alpha(x)^2$.) Let R^+ be the subring of elements of norm at most 1. Again, let $\mathbb{B}_{R,+}$ be the Fréchet completion of $W(R^+)[p^{-1}]$ for $\lambda(\alpha^r)$ for all $r > 0$. Define $\mathcal{M}(R)$ by glueing: take the Gel'fand spectrum after Fréchet completing for r in a closed interval, then take the union over intervals.

If R is a Banach algebra over an analytic field with nontrivial norm, then the action of φ^* on $\mathcal{M}(R)$ is totally discontinuous. Quotienting gives a homotopy circle bundle over $\mathcal{M}(R)$. For $R = F$, this acts like an *analytic skeleton* of the Fargues-Fontaine curve.

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Étale covers and local systems

Let R be a perfect uniform Banach F-algebra. Put

$$P_R = \bigoplus_{n=0}^{\infty} P_{R,n}, \qquad P_{R,n} = \mathbb{B}_{R,+}^{\varphi=p^n}.$$

Now $t \in P_1$ cuts out a closed subscheme of $\mathrm{Proj}(P)$ whose residue ring $\tilde{R}$ is a Banach algebra over the completion $\mathbb{Q}_p(\mu_{p^\infty})$.

Theorem (K-Liu, Scholze; after Faltings, Andreatta, Gabber-Ramero)

There is a natural equivalence between finite étale R-algebras and finite étale $\tilde{R}$-algebras.

Theorem (K-Liu)

The categories of étale $\mathbb{Q}_p$-local systems on $\mathcal{M}(R)$, étale $\mathbb{Q}_p$-local systems on $\mathcal{M}(\tilde{R})$, and **fibrewise semistable** *vector bundles of degree 0 on* $\mathrm{Proj}(P_R)$ *are naturally (in R) equivalent.*

Deeply ramified covers

Let A be an affinoid algebra over $\mathbb{Q}_p$. To describe local systems on $\mathcal{M}(A)$ using the previous theorem, we make a *deeply ramified* extension $\tilde{A}$ of A which has the form $\tilde{R}$ for some perfect Banach F-algebra R, equipped with a Galois action which can be used to specify descent data. It is sufficient to ensure that Frobenius on $\tilde{A}^+/(p)$ is surjective.

For instance, if $\mathcal{M}(A)$ embeds into an affine space with coordinates $T_1, \dots, T_n$, we can form

$$\tilde{A} = A \widehat{\otimes}_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})[T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$$

(with some care if the T_i are not units in A).

One can also make universal constructions using suitable sites, e.g., Scholze's *pro-étale site*. The latter is best suited for proving a *relative comparison isomorphism* between étale and de Rham cohomology.

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What's easy: containment of p

It is desirable to force all dependence on the prime p in these constructions through the p-adic absolute value, avoiding use of algebraic properties (e.g., the Frobenius map in characteristic p). Here are some easy ways to move in this direction.

- Adjoin all roots of unity and the T_i, not just the p-power one. This is still compatible with use of the pro-étale site. (For the original Fargues-Fontaine curve, one replaces P by some sort of product over p-adic valuations on $\mathbb{Q}^{\mathrm{ab}}$.)
- Work with $W(R)$ instead of R, as this can be reconstructed directly from $W(\tilde{R})$ by taking the inverse limit under Frobenius (Davis-K).
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What's harder

- After making the changes suggested on the previous slide, can one consider an archimedean Banach algebra? And can one say anything meaningful about ordinary Hodge theory?
- Now consider an "adelic Banach algebra". Can one imitate Scholze's relative comparison theory to define "a coefficient object on the BC-system associated to a smooth proper $\mathbb{Q}$-scheme"? Can one get back to de Rham cohomology or étale cohomology?
- What do K-theory and the de Rham-Witt complex have to do with this?
- What exactly is the arithmetic curve? How does one associate cohomology to its coefficient objects so as to give spectral interpretations of L-functions?

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