Relative $p$-adic Hodge theory and Rapoport-Zink period domains

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What is Hodge theory?

To an algebraic variety $X$ over $\mathbb{C}$, we can associate Betti cohomology $H^i_{\text{Betti}}(X, \mathbb{R})$ and de Rham cohomology $H^i_{\text{dR}}(X, \mathbb{C})$. These $\mathbb{R}$-vector spaces are canonically isomorphic, but carry different additional structures: Betti cohomology carries an integral lattice (the image of cohomology with $\mathbb{Z}$-coefficients) while de Rham cohomology carries a complex structure and a decreasing filtration (the Hodge filtration).

One defines a Hodge structure to be a finite-dimensional $\mathbb{R}$-vector space equipped with a $\mathbb{Z}$-lattice, a complex structure, and a decreasing filtration. Hodge theory is the study of such objects and how they can arise from the cohomology of varieties, individually and in families.
What is $p$-adic Hodge theory?

Let $K_0$ be a finite unramified extension of $\mathbb{Q}_p$, and let $K$ be a finite totally ramified extension of $K_0$. Let $\mathfrak{o}_K, k$ be the valuation ring and residue field of $K$. Let $\overline{K}$ be an algebraic closure of $K$.

To a smooth proper scheme $X$ over $\mathfrak{o}_K$, we can associate:

- étale cohomology $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_p)$, which carries an action of the absolute Galois group $G_K$;
- de Rham cohomology $H^i_{\text{dR}}(X_K, K)$, which carries a Hodge filtration;
- rigid cohomology $H^i_{\text{rig}}(X_k, K_0)$, which carries a Frobenius map $\varphi$.

The relationship among these structures is the concern of $p$-adic Hodge theory.
Filtered isocrystals

Since $K_0$ is unramified, it carries a unique automorphism $\varphi$ lifting the $p$-th power (Frobenius) map mod $p$. For any field $L$ containing $K_0$, a filtered isocrystal over $L$ (with respect to $K_0$) consists of

- a finite-dimensional $K_0$-vector space $D$ equipped with a semilinear bijective $\varphi$-action (an isocrystal);
- an exhaustive decreasing filtration $\text{Fil}^\bullet$ on $D_L = D \otimes_{K_0} L$.

We get such an object over $K$ from the canonical comparison isomorphism

$$H^{i}_{dR}(X_K, K) \cong H^{i}_{\text{rig}}(X_k, K_0) \otimes_{K_0} K.$$ 

Consequently, filtered isocrystals may be viewed as $p$-adic Hodge structures.
The relationship between de Rham and étale cohomologies is subtler. Following a suggestion of Grothendieck, Fontaine proposed and Faltings confirmed the existence of a comparison isomorphism

\[ H^i_{\text{et}}(X_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{crys}} \cong H^i_{\text{dR}}(X_K, K) \otimes_K B_{\text{crys}} \]

for a certain topological $K$-algebra $B_{\text{crys}}$ equipped with a $G_K$-action, a Frobenius action, and a filtration (the *ring of crystalline periods*).

The resulting module over $B_{\text{crys}}$ carries a $G_K$-action, a Frobenius action, and a filtration. One recovers étale cohomology by taking the $\varphi$-invariants of $\text{Fil}^0$. One recovers de Rham and rigid cohomologies by taking Galois invariants.
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Let $D$ be a filtered isocrystal over $L$ with respect to $K_0$. For any basis of $D$, the action of $\varphi$ is specified by a matrix $A$; changing basis by the matrix $U$ replaces $A$ by $U^{-1}A\varphi(U)$. The $p$-adic valuation of $\det(A)$ is thus independent of the choice of basis; it is called the \textit{Frobenius degree} of $D$.

The \textit{Hodge-Tate weights} of $D$ are the integers $i$ with $\text{Fil}^i \neq \text{Fil}^{i+1}$, counted with multiplicity $\dim(\text{Fil}^i / \text{Fil}^{i+1})$. In the case of de Rham cohomology, these compute Hodge numbers.
Weak admissibility for filtered isocrystals

Let $D$ be a filtered isocrystal over $L$ with respect to $K_0$. Let $t_N(D)$ be the Frobenius degree of $D$. Let $t_H(D)$ be the sum of the Hodge-Tate weights of $D$. Put $\deg(D) = t_H(D) - t_N(D)$.

By analogy with semistability for vector bundles, $D$ is weakly admissible if:

(a) $\deg(D) = 0$; and
(b) for each $\varphi$-stable subspace $D'$ of $D$ (over $K_0$), $\deg(D') \leq 0$.

This holds whenever $D$ arises from cohomology. More generally, if $D$ is weakly admissible, there exist a continuous representation of $G_K$ on a finite-dimensional $\mathbb{Q}_p$-vector space $V$ and an isomorphism

$$V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\text{crys}} \cong D \otimes_{K_0} \mathbb{B}_{\text{crys}}$$

respecting extra structures (Colmez-Fontaine). Such representations are said to be crystalline.
Interpolating Galois representations

We may view $V$ as a local system in finite-dimensional $\mathbb{Q}_p$-vector spaces on Spec $K$ for the étale topology, i.e., as a étale $\mathbb{Q}_p$-local system.

Let $D$ be an isocrystal over $K_0$, and let $H$ be a finite multiset of integers. Let $\mathcal{F}_{D,H}$ be the moduli space of filtrations on $D$ with weights $H$; it is a partial flag variety. Each weakly admissible closed point of $\mathcal{F}_{D,H}$ gives rise to a crystalline representation.

Problem (Rapoport-Zink)

Interpolate these representations by an étale $\mathbb{Q}_p$-local system on some analytic subspace of $\mathcal{F}_{D,H}$ (in a sense to be clarified shortly).

In some cases, this space receives a period morphism from a certain deformation space; e.g., for weights in $\{0, 1\}$, there is a period morphism from the deformation space of a suitable $p$-divisible group or abelian variety (as in ordinary Hodge theory!).
Analytic spaces over nonarchimedean fields

There are multiple notions of analytic spaces over $K_0$. In all cases, the basic objects arise from *affinoid algebras*, the Banach algebras receiving surjections from a Tate algebra (the completion of some $K_0[T_1, \ldots, T_n]$ for the Gauss norm).

Tate’s *rigid analytic spaces* are formed from spectra of maximal ideals, glued using Grothendieck topologies. These spaces carry totally disconnected ordinary topologies.

We use Berkovich’s *nonarchimedean analytic spaces*, formed from Gel’fand spectra, i.e., spaces of bounded multiplicative seminorms. These carry locally path-connected topologies, and are inverse limits of finite polyhedral complexes (hence “tropical”).

Each point in a Berkovich space has a *residue field*, which is complete but (unlike in Tate’s theory) possibly infinite over $K_0$. 
The admissible locus and its local system

For fixed isocrystal $D$ and weights $H$, the moduli space of weakly admissible filtrations on $D$ (defined over complete field extensions of $K$) is an open subspace $\mathcal{F}_{D,H}^{wa}$ of $\mathcal{F}_{D,H}$, the weakly admissible locus.

**Theorem**

There exist an open subspace $\mathcal{F}_{D,H}^a$ of $\mathcal{F}_{D,H}^{wa}$ (the admissible locus) containing all weakly admissible closed points of $\mathcal{F}_{D,H}$, and an étale $\mathbb{Q}_p$-local system on $\mathcal{F}_{D,H}^a$ specializing to the associated crystalline representation at each weakly admissible closed point.

In general $\mathcal{F}_{D,H}^a \neq \mathcal{F}_{D,H}^{wa}$. The admissible locus is defined using a related but subtler notion of semistability, using vector bundles with Frobenius actions (analogues of Drinfel’d’s shtukas).

For varying $D$, the situation is only partly understood (Pappas-Rapoport, Hellmann). One must replace Berkovich spaces with Huber’s adic spaces.
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Semistability over the Robba ring

The Robba ring $\mathcal{R}$ consists of formal Laurent series $\sum_{i=-\infty}^{\infty} c_i \pi^i$ over $K_0$ which converge in some annulus $* \leq |\pi| < 1$. Extend the Frobenius $\varphi$ on $K_0$ to $\mathcal{R}$ so that $\varphi(\pi) = (1 + \pi)^p - 1$.

A $\varphi$-module over $\mathcal{R}$ is a finite free module $M$ plus an isomorphism $\varphi^* M \cong M$. Since units of $\mathcal{R}$ have bounded coefficients and hence $p$-adic valuations, we may again define Frobenius degree as the $p$-adic valuation of the matrix via which $\varphi$ acts on a basis.

$M$ is étale if on some basis, $\varphi$ acts via a matrix $U$ for which $U, U^{-1}$ have all coefficients in $\mathfrak{o}_{K_0}$ (the valuation ring of $K_0$).

Theorem (K, 2004)

A $\varphi$-module over $\mathcal{R}$ of Frobenius degree 0 is étale if and only if it is semistable (admits no $\varphi$-stable submodule of negative Frobenius degree).
\((\varphi, \Gamma)\)-modules

The group \(\Gamma = \mathbb{Z}_p^\times\) acts \(K_0\)-linearly on \(R\) with \(\gamma \in \Gamma\) acting as \(\pi \mapsto (1 + \pi)^\gamma - 1\). An (étale) \((\varphi, \Gamma)\)-module is an (étale) \(\varphi\)-module \(M\) plus a (continuous) semilinear \(\Gamma\)-action commuting with \(\varphi\).

Theorem (Fontaine, Cherbonnier-Colmez, Berger, K, etc.)

The category of étale \((\varphi, \Gamma)\)-modules is equivalent to the category of continuous representations of \(G_{K_0}\) on finite-dimensional \(\mathbb{Q}_p\)-vector spaces.

Sketch of the passage from \((\varphi, \Gamma)\)-modules to representations:

- construct \(\varphi\)-invariant sections over a pro-étale cover of an annulus;
- restrict to \(\pi = \epsilon_n - 1\) for \(\epsilon_n\) a primitive \(p^n\)-th root of 1 for some large \(n\), producing a representation of \(G_K(\epsilon_n)\);
- use \(\Gamma\) to descend to a representation of \(G_K\).

A similar result holds for \(K\) finite over \(K_0\) (replacing \(R\) with a certain finite étale extension).
Let $D$ be a filtered isocrystal over $K_0$. Extend $D$ trivially to a vector bundle $V_D$ over the open unit disc over $K_0$, carrying semilinear actions of $\varphi$ and $\Gamma$ (with $\Gamma$ acting trivially on $D$).

Modify $V_D$ at 0 to change the $\pi$-adic filtration on $V_D \otimes K_0((\pi))$ by tensoring with the provided filtration on $D$. By pullback by $\varphi$, move this modification to $\pi = \zeta - 1$ for each $p$-power root of unity $\zeta$. Glue to obtain a vector bundle $V'_D$ plus an isomorphism $\varphi^*V'_D \cong V_D$ away from $\pi = 0$.

**Theorem (Berger)**

The $(\varphi, \Gamma)$-module $M_D = V'_D \otimes \mathcal{R}$ is étale iff $D$ is weakly admissible. If so, $M_D$ gives rise to the crystalline representation associated to $D$.

Again, a similar construction works for $K$ finite over $K_0$. 
Illustration

First modify $M_D$ here...

...then pull back to copy this modification.

\[ \pi = \epsilon_0 - 1 = 0 \]

\[ \pi = \epsilon_1 - 1 \]

\[ \pi = \epsilon_2 - 1 \]

\[ \pi = \epsilon_3 - 1 \]
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Fix an isocrystal $D$ over $K_0$ and some weights $H$. We work on a piece of $\mathcal{F}_{D,H}$ isomorphic to a closed unit polydisc, i.e., the analytic space associated to a Tate algebra in the variables $T_1, \ldots, T_n$. To globalize, use glueing for relative $(\varphi, \Gamma)$-modules (described in my ICM satellite lecture).

We work on $X$, the product of the closed polydisc in $T_1, \ldots, T_n$ with the open $\pi$-disc. The analytic functions on $X$ are power series in $T_1, \ldots, T_n, \pi$; we extend the Frobenius lift $\varphi$ to this ring by

$$\pi \mapsto (1 + \pi)^p - 1, \quad T_i \mapsto T_i^p.$$ 

For $J = \{1, \ldots, n\}$, we also have an action of $\Gamma_J = \mathbb{Z}_p^\times \times \mathbb{Z}_p^J$ generated by the action of $\Gamma = \mathbb{Z}_p^\times$ as before, plus the substitutions $T_i \mapsto (1 + \pi) T_i$. 
Define a trivial vector bundle on $X$ with fibres $D$, with semilinear actions of $\varphi$ and $\Gamma_J$ (again with $\Gamma_J$ fixing $D$).

As before, we modify along $\pi = 0$ to change the $\pi$-adic filtration, but now using the universal filtration from $\mathcal{F}_{D,H}$. Again, pull back using $\varphi$ to $\pi = \zeta - 1$ for each $p$-power root of unity $\zeta$. Glueing now gives a $(\varphi, \Gamma_J)$-module $M_D$ on $X$.

The goal: construct an open subspace $V$ of $X$ on which $M_D$ is “étale”. In particular, $M_D$ will admit $\varphi$-invariant sections on some pro-étale cover of $V$; restricting to $\pi = 0$ will cut out a subspace of $\mathcal{F}_{D,H}$ on which we obtain a local system. (See my ICM satellite lecture for more discussion.)

We’ll use *Witt vectors* for this. In a similar spirit, see recent work on Fargues-Fontaine on foundations of $p$-adic Hodge theory.
Witt vectors and their geometry

Theorem (Teichmüller, Witt)

For $R$ a perfect ring in which $p = 0$ (i.e., Frobenius is bijective), there is a unique $p$-adically separated and complete ring $W(R)$ with $W(R)/(p) \cong R$. There is also a multiplicative map $[\bullet] : R \rightarrow W(R)$, using which each $x \in W(R)$ can be written uniquely as $\sum_{i=0}^{\infty} p^i [\bar{x}_i]$ with $\bar{x}_i \in R$.

Although $[\bullet]$ is not a ring homomorphism, restriction along $[\bullet]$ does define a map $\mu : \mathcal{M}(W(R)) \rightarrow \mathcal{M}(R)$, where $\mathcal{M}(A)$ denotes the space of bounded multiplicative seminorms on a commutative Banach ring $A$. (Here $R$ carries the trivial norm and $W(R)$ the $p$-adic norm.)

A one-sided inverse to $\mu$ is given by $\lambda : \mathcal{M}(R) \rightarrow \mathcal{M}(W(R))$:

$$\lambda(\alpha)(\sum_i p^i [\bar{x}_i]) = \max_i \{p^{-i} \alpha(\bar{x}_i)\}.$$

The maps $\lambda$ and $\mu$ induce a homotopy equivalence!
Construction of a crystalline local system

Interpolation using Witt vectors

The inverse limit of
\[ \cdots \xrightarrow{\varphi} X \xrightarrow{\varphi} X \]
embeds into \( \mathcal{M}(W(R)) \) for \( R = k[\overline{T}_1, \ldots, \overline{T}_n][[\overline{\pi}]]^{\text{perf}} \).

**Theorem**

*There is an open analytic subspace \( V \) of \( X \) which is the projection of*

\[ \bigcup \{ \mu^{-1}(\alpha^s) : \alpha \in U, s > 0 \} \]

*for some open subspace \( U \) of \( \mathcal{M}(R) \), such that \( V \cap \{ \pi = 0 \} \) contains all weakly admissible closed points, and \( M_D \) is étale on \( V \).*

We initially get a slightly smaller set (concentrated near the boundary) on which \( \Gamma_J \) acts. The \( \Gamma_J \)-action then provides descent data, so that we can replace \( V \) by its \( \varphi^* \)-image to grow \( V \) towards \( \pi = 0 \).
Illustration

\[ \pi = \epsilon_3 - 1 \]
\[ \pi = \epsilon_2 - 1 \]
\[ \pi = \epsilon_1 - 1 \]
\[ \pi = \epsilon_0 - 1 = 0 \]

We show \( M_D \) is étale on \( V \) above the dashed line. We then use \( \Gamma_J \) to descend along \( \varphi^* \), and get a local system over the bottom gray region. Here \( \epsilon_n \) is a primitive \( p^n \)-th root of 1.