Relative p-adic Hodge theory and Rapoport-Zink period domains

Kiran S. Kedlaya

Department of Mathematics, Massachusetts Institute of Technology
Department of Mathematics, University of California, San Diego
kedlaya@(mit|ucsd).edu

International Congress of Mathematicians
Hyderabad International Conference Centre, August 20, 2010

Work in progress with Ruochuan Liu; inconvenience regretted. Slides available at http://math.mit.edu/~kedlaya/papers/talks.shtml; see there also for slides from my related ICM satellite lecture.

Supported by NSF, DARPA, MIT, IAS.

Contents

Contents

What is Hodge theory?

To an algebraic variety X over $\mathbb{C}$, we can associate *Betti cohomology* $H^i_{\mathrm{Betti}}(X, \mathbb{R})$ and *de Rham cohomology* $H^i_{\mathrm{dR}}(X, \mathbb{C})$. These $\mathbb{R}$-vector spaces are canonically isomorphic, but carry different additional structures: Betti cohomology carries an integral lattice (the image of cohomology with $\mathbb{Z}$-coefficients) while de Rham cohomology carries a complex structure and a decreasing filtration (the *Hodge filtration*).

One defines a *Hodge structure* to be a finite-dimensional $\mathbb{R}$-vector space equipped with a $\mathbb{Z}$-lattice, a complex structure, and a decreasing filtration. *Hodge theory* is the study of such objects and how they can arise from the cohomology of varieties, individually and in families.

What is p-adic Hodge theory?

Let K_0 be a finite unramified extension of $\mathbb{Q}_p$, and let K be a finite totally ramified extension of K_0. Let $\mathfrak{o}_K, k$ be the valuation ring and residue field of K. Let $\overline{K}$ be an algebraic closure of K.

To a smooth proper scheme X over $\mathfrak{o}_K$, we can associate:

- *étale cohomology* $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$, which carries an action of the absolute Galois group G_K;
- *de Rham cohomology* $H^i_{\mathrm{dR}}(X_K, K)$, which carries a *Hodge filtration*;
- *rigid cohomology* $H^i_{\mathrm{rig}}(X_k, K_0)$, which carries a *Frobenius map* φ.

The relationship among these structures is the concern of *p-adic Hodge theory*.

Filtered isocrystals

Since K_0 is unramified, it carries a unique automorphism φ lifting the p-th power (Frobenius) map mod p. For any field L containing K_0, a *filtered isocrystal* over L (with respect to K_0) consists of

- a finite-dimensional K_0-vector space D equipped with a semilinear bijective φ-action (an *isocrystal*);
- an exhaustive decreasing filtration $\mathrm{Fil}^\bullet$ on $D_L = D \otimes_{K_0} L$.

We get such an object over K from the canonical comparison isomorphism

$$H^i_{\mathrm{dR}}(X_K, K) \cong H^i_{\mathrm{rig}}(X_k, K_0) \otimes_{K_0} K.$$

Consequently, filtered isocrystals may be viewed as *p-adic Hodge structures*.

The mysterious functor

The relationship between de Rham and étale cohomologies is subtler. Following a suggestion of Grothendieck, Fontaine proposed and Faltings confirmed the existence of a comparison isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}} \cong H^i_{\mathrm{dR}}(X_K, K) \otimes_K \mathbf{B}_{\mathrm{crys}}$$

for a certain topological K-algebra $\mathbf{B}_{\mathrm{crys}}$ equipped with a G_K-action, a Frobenius action, and a filtration (the *ring of crystalline periods*).

The resulting module over $\mathbf{B}_{\mathrm{crys}}$ carries a G_K-action, a Frobenius action, and a filtration. One recovers étale cohomology by taking the φ-invariants of Fil^0. One recovers de Rham and rigid cohomolgies by taking Galois invariants.

Contents

Numerical invariants

Let D be a filtered isocrystal over L with respect to K_0. For any basis of D, the action of φ is specified by a matrix A; changing basis by the matrix U replaces A by $U^{-1}A\varphi(U)$. The p-adic valuation of $\det(A)$ is thus independent of the choice of basis; it is called the *Frobenius degree* of D.

The *Hodge-Tate weights* of D are the integers i with $\mathrm{Fil}^i \neq \mathrm{Fil}^{i+1}$, counted with multiplicity $\dim(\mathrm{Fil}^i / \mathrm{Fil}^{i+1})$. In the case of de Rham cohomology, these compute Hodge numbers.

Weak admissibility for filtered isocrystals

Let D be a filtered isocrystal over L with respect to K_0. Let $t_N(D)$ be the Frobenius degree of D. Let $t_H(D)$ be the sum of the Hodge-Tate weights of D. Put $\deg(D) = t_H(D) - t_N(D)$.

By analogy with semistability for vector bundles, D is *weakly admissible* if:

(a) $\deg(D) = 0$; and

(b) for each φ-stable subspace D' of D (over K_0), $\deg(D') \leq 0$.

This holds whenever D arises from cohomology. More generally, if D is weakly admissible, there exist a continuous representation of G_K on a finite-dimensional $\mathbb{Q}_p$-vector space V and an isomorphism

$$V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}} \cong D \otimes_{K_0} \mathbf{B}_{\mathrm{crys}}$$

respecting extra structures (Colmez-Fontaine). Such representations are said to be *crystalline*.

Interpolating Galois representations

We may view V as a local system in finite-dimensional $\mathbb{Q}_p$-vector spaces on $\operatorname{Spec} K$ for the étale topology, i.e., as a *étale $\mathbb{Q}_p$-local system*.

Let D be an isocrystal over K_0, and let H be a finite multiset of integers. Let $\mathcal{F}_{D,H}$ be the moduli space of filtrations on D with weights H; it is a partial flag variety. Each weakly admissible closed point of $\mathcal{F}_{D,H}$ gives rise to a crystalline representation.

Problem (Rapoport-Zink)

Interpolate these representations by an étale $\mathbb{Q}_p$-local system on some analytic subspace of $\mathcal{F}_{D,H}$ (in a sense to be clarified shortly).

In some cases, this space receives a *period morphism* from a certain deformation space; e.g., for weights in $\{0, 1\}$, there is a period morphism from the deformation space of a suitable p-divisible group or abelian variety (as in ordinary Hodge theory!).

Analytic spaces over nonarchimedean fields

There are multiple notions of analytic spaces over K_0. In all cases, the basic objects arise from *affinoid algebras*, the Banach algebras receiving surjections from a Tate algebra (the completion of some $K_0[T_1, \dots, T_n]$ for the Gauss norm).

Tate's *rigid analytic spaces* are formed from spectra of maximal ideals, glued using Grothendieck topologies. These spaces carry totally disconnected ordinary topologies.

We use Berkovich's *nonarchimedean analytic spaces*, formed from *Gel'fand spectra*, i.e., spaces of bounded multiplicative seminorms. These carry locally path-connected topologies, and are inverse limits of finite polyhedral complexes (hence "tropical").

Each point in a Berkovich space has a *residue field*, which is complete but (unlike in Tate's theory) possibly infinite over K_0.

The admissible locus and its local system

For fixed isocrystal D and weights H, he moduli space of weakly admissible filtrations on D (defined over complete field extensions of K) is an open subspace $\mathcal{F}^{wa}_{D,H}$ of $\mathcal{F}_{D,H}$, the *weakly admissible locus*.

Theorem

There exist an open subspace $\mathcal{F}^{a}_{D,H}$ of $\mathcal{F}^{wa}_{D,H}$ (the **admissible locus***) containing all weakly admissible closed points of $\mathcal{F}_{D,H}$, and an étale $\mathbb{Q}_p$-local system on $\mathcal{F}^{a}_{D,H}$ specializing to the associated crystalline representation at each weakly admissible closed point.*

In general $\mathcal{F}^{a}_{D,H} \neq \mathcal{F}^{wa}_{D,H}$. The admissible locus is defined using a related but subtler notion of semistability, using vector bundles with Frobenius actions (analogues of Drinfel'd's *shtukas*).

For varying D, the situation is only partly understood (Pappas-Rapoport, Hellmann). One must replace Berkovich spaces with Huber's *adic spaces*.

Contents

Semistability over the Robba ring

The *Robba ring* $\mathcal{R}$ consists of formal Laurent series $\sum_{i=-\infty}^{\infty} c_i \pi^i$ over K_0 which converge in some annulus $* \leq |\pi| < 1$. Extend the Frobenius φ on K_0 to $\mathcal{R}$ so that $\varphi(\pi) = (1+\pi)^p - 1$.

A *φ-module* over $\mathcal{R}$ is a finite free module M plus an isomorphism $\varphi^* M \cong M$. Since units of $\mathcal{R}$ have bounded coefficients and hence p-adic valuations, we may again define *Frobenius degree* as the p-adic valuation of the matrix via which φ acts on a basis.

M is *étale* if on some basis, φ acts via a matrix U for which U, U^{-1} have all coefficients in $\mathfrak{o}_{K_0}$ (the valuation ring of K_0).

Theorem (K, 2004)

A φ-module over $\mathcal{R}$ of Frobenius degree 0 is étale if and only if it is semistable (admits no φ-stable submodule of negative Frobenius degree).

(φ, Γ)-modules

The group $\Gamma = \mathbb{Z}_p^\times$ acts K_0-linearly on $\mathcal{R}$ with $\gamma \in \Gamma$ acting as $\pi \mapsto (1+\pi)^\gamma - 1$. An *(étale) (φ, Γ)-module* is an (étale) φ-module M plus a (continuous) semilinear Γ-action commuting with φ.

Theorem (Fontaine, Cherbonnier-Colmez, Berger, K, etc.)

The category of étale (φ, Γ)-modules is equivalent to the category of continuous representations of G_{K_0} on finite-dimensional $\mathbb{Q}_p$-vector spaces.

Sketch of the passage from (φ, Γ)-modules to representations:

- construct φ-invariant sections over a pro-étale cover of an annulus;
- restrict to $\pi = \epsilon_n - 1$ for ϵ_n a primitive p^n-th root of 1 for some large n, producing a representation of $G_{K(\epsilon_n)}$;
- use Γ to descend to a representation of G_K.

A similar result holds for K finite over K_0 (replacing $\mathcal{R}$ with a certain finite étale extension).

The nonmysterious functor: isocrystals to representations

Let D be a filtered isocrystal over K_0. Extend D trivially to a vector bundle V_D over the open unit disc over K_0, carrying semilinear actions of φ and Γ (with Γ acting trivially on D).

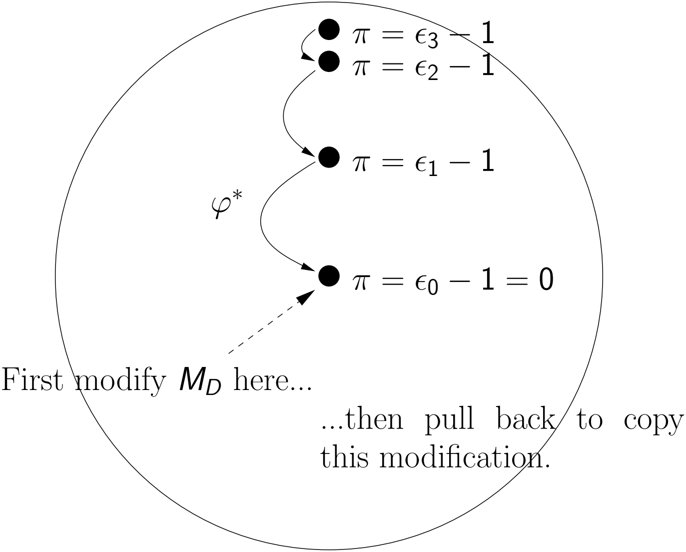
Modify V_D at 0 to change the π-adic filtration on $V_D \otimes K_0((\pi))$ by tensoring with the provided filtration on D. By pullback by φ, move this modification to $\pi = \zeta - 1$ for each p-power root of unity ζ. Glue to obtain a vector bundle V'_D plus an isomorphism $\varphi^* V'_D \cong V_D$ away from $\pi = 0$.

Theorem (Berger)

The (φ, Γ)-module $M_D = V'_D \otimes \mathcal{R}$ is étale iff D is weakly admissible. If so, M_D gives rise to the crystalline representation associated to D.

Again, a similar construction works for K finite over K_0.

Illustration



Contents

Relative setup

Fix an isocrystal D over K_0 and some weights H. We work on a piece of $\mathcal{F}_{D,H}$ isomorphic to a closed unit polydisc, i.e., the analytic space associated to a Tate algebra in the variables $T_1, \dots, T_n$. To globalize, use glueing for relative (φ, Γ)-modules (described in my ICM satellite lecture).

We work on X, the product of the closed polydisc in $T_1, \dots, T_n$ with the open π-disc. The analytic functions on X are power series in $T_1, \dots, T_n, \pi$; we extend the Frobenius lift φ to this ring by

$$\pi \mapsto (1+\pi)^p - 1, \qquad T_i \mapsto T_i^p.$$

For $J = \{1, \dots, n\}$, we also have an action of $\Gamma_J = \mathbb{Z}_p^\times \ltimes \mathbb{Z}_p^J$ generated by the action of $\Gamma = \mathbb{Z}_p^\times$ as before, plus the substitutions $T_i \mapsto (1+\pi)T_i$.

Modification

Define a trivial vector bundle on X with fibres D, with semilinear actions of φ and Γ_J (again with Γ_J fixing D).

As before, we modify along $\pi = 0$ to change the π-adic filtration, but now using the universal filtration from $\mathcal{F}_{D,H}$. Again, pull back using φ to $\pi = \zeta - 1$ for each p-power root of unity ζ. Glueing now gives a (φ, Γ_J)-*module* M_D on X.

The goal: construct an open subspace V of X on which M_D is "étale". In particular, M_D will admit φ-invariant sections on some pro-étale cover of V; restricting to $\pi = 0$ will cut out a subspace of $\mathcal{F}_{D,H}$ on which we obtain a local system. (See my ICM satellite lecture for more discussion.)

We'll use *Witt vectors* for this. In a similar spirit, see recent work on Fargues-Fontaine on foundations of p-adic Hodge theory.

Witt vectors and their geometry

Theorem (Teichmüller, Witt)

For R a perfect ring in which $p = 0$ (i.e., Frobenius is bijective), there is a unique p-adically separated and complete ring $W(R)$ with $W(R)/(p) \cong R$. There is also a multiplicative map $[\bullet] : R \to W(R)$, using which each $x \in W(R)$ can be written uniquely as $\sum_{i=0}^{\infty} p^i [\overline{x}_i]$ with $\overline{x}_i \in R$.

Although $[\bullet]$ is not a ring homomorphism, restriction along $[\bullet]$ does define a map $\mu : \mathcal{M}(W(R)) \to \mathcal{M}(R)$, where $\mathcal{M}(A)$ denotes the space of bounded multiplicative seminorms on a commutative Banach ring A. (Here R carries the trivial norm and $W(R)$ the p-adic norm.)

A one-sided inverse to μ is given by $\lambda : \mathcal{M}(R) \to \mathcal{M}(W(R))$:

$$\lambda(\alpha)(\sum_i p^i [\overline{x}_i]) = \max_i \{p^{-i} \alpha(\overline{x}_i)\}.$$

The maps λ and μ induce a homotopy equivalence!

Interpolation using Witt vectors

The inverse limit of

$$\cdots \xrightarrow{\varphi} X \xrightarrow{\varphi} X$$

embeds into $\mathcal{M}(W(R))$ for $R = k[\overline{T}_1, \dots, \overline{T}_n][\![\overline{\pi}]\!]^{\mathrm{perf}}$.

Theorem

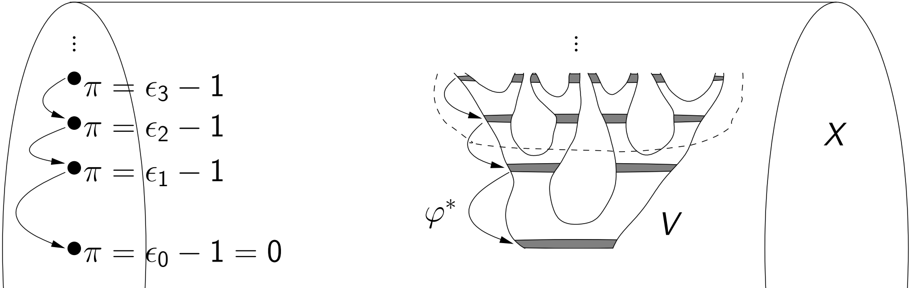
There is an open analytic subspace V of X which is the projection of

$$\bigcup \{\mu^{-1}(\alpha^s) : \alpha \in U, s > 0\}$$

for some open subspace U of $\mathcal{M}(R)$, such that $V \cap \{\pi = 0\}$ contains all weakly admissible closed points, and M_D is étale on V.

We initially get a slightly smaller set (concentrated near the boundary) on which Γ_J acts. The Γ_J-action then provides descent data, so that we can replace V by its φ^*-image to grow V towards $\pi = 0$.

Illustration



We show M_D is étale on V above the dashed line. We then use Γ_J to descend along φ^*, and get a local system over the bottom gray region. Here ϵ_n is a primitive p^n-th root of 1.