Sato-Tate groups of higher weight motives

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Explicit Methods for Modularity of K3 Surfaces and
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For a motive M (with $\mathbb{Q}$-coefficients), consider its L-function in the analytic normalization:

$$L(s) = \prod_p L_p(s) = \prod_p F_p(p^{-s})^{-1}, \quad F_p(T) = 1 - a_p T + \cdots .$$

Conjecture (generalized Sato-Tate conjecture; Serre, 1994)

The polynomials $F_p(T)$ are equidistributed for the image of Haar measure (via the characteristic polynomial map) on a specified compact Lie group $\mathrm{ST}(M)$ *(the* **Sato-Tate group***).*

E.g,, the a_p vary like traces of random matrices in $\mathrm{ST}(M)$.

Proposition

For any given degree, weight, and Hodge numbers (i.e., Gamma factors), there are only finitely many possible Sato-Tate groups.

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Example: elliptic curves

Take $M = H^1(E)$ with E an elliptic curve over $\mathbb{Q}$.

If E has CM, then $\mathrm{ST}(M)$ is the normalizer of $\mathrm{SO}(2, \mathbb{R})$ in $\mathrm{SU}(2)$:

`http://math.mit.edu/~drew/g1_D2_a1f.gif`.

Equidistribution follows easily from CM theory (Hecke).

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If we consider E over a number field K, then the CM picture changes if the CM field is contained in K, as $\mathrm{ST}(M)$ decreases to $\mathrm{SO}(2, \mathbb{R})$:

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More examples to consider

For the rest of the talk, we will be interested in the following three classes of motives. Here K denotes an arbitrary number field (but you may assume $K = \mathbb{Q}$), w is the motivic weight, $(h^{0,w}, \dots, h^{w,0})$ is the Hodge vector, and $d = \sum_{p+q=w} h^{p,q}$ is the degree of the associated L-function.

- M has weight 1 and Hodge vector (g, g). This means that $M = H^1(A)$ for A/K an abelian variety of dimension g.
- M has weight 2 and Hodge vector $(1, 20, 1)$. In particular, we want[1] $M = H^2(X)$ for X/K a K3 surface.
- M has weight 3 and Hodge vector $(1, 1, 1, 1)$, e.g., a hypergeometric motive from the Dwork pencil

$$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 = \lambda x_0 x_1 x_2 x_3 x_4.$$

[1] To force this, we must fix some extra data, e.g., the intersection pairing and the ample cone.

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The Betti-Hodge realization and the Mumford-Tate group

Fix an embedding $K \hookrightarrow \mathbb{C}$. Let V denote the Betti (singular) cohomology of M with $\mathbb{Q}$-coefficients; then $\dim_{\mathbb{Q}} V = d$.

The duality $M \times M \to \mathbb{Q}(-w)$ induces a perfect bilinear pairing ψ on V. Let $\mathrm{GIso}(V, \psi)$ be the associated group of symplectic (if w is odd) or orthogonal (if w is even) similitudes.

The space $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ admits a canonical *Hodge decomposition* $\bigoplus_{p+q=w} V^{p,q}$ with $\dim_{\mathbb{C}} V^{p,q} = h^{p,q}$. Let

$$\mu_{\infty,V} : \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}})$$

be the cocharacter acting with weight $-p$ on $V^{p,q}$.

The *Mumford-Tate group* of M is the minimal (connected) $\mathbb{Q}$-algebraic subgroup $\mathrm{MT}(M)$ of $\mathrm{GIso}(V, \psi)$ through which $\mu_{\infty,V}$ factors.

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The *Mumford-Tate group* of M is the minimal (connected) $\mathbb{Q}$-algebraic subgroup $\mathrm{MT}(M)$ of $\mathrm{GIso}(V, \psi)$ through which $\mu_{\infty,V}$ factors.

For n a positive integer for which wn is even, put $p = wn/2$ and

$$(V^{\otimes n})^{p,p} := (V_{\mathbb{C}}^{\otimes n})^{p,p} \cap V^{\otimes n}.$$

Then $\mathrm{MT}(M)$ can also be characterized as the maximal subgroup of $\mathrm{GIso}(V, \psi)$ fixing $(V^{\otimes n})^{p,p}$ for all n.

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The motivic Galois group

Under the Hodge conjecture[2], $(V^{\otimes n})^{p,p}$ is spanned by the Chern classes of algebraic cycles defined over $\overline{K}$. We thus have an action of the absolute Galois group G_K on $(V^{\otimes n})^{p,p}$.

The *motivic Galois group* $\mathrm{Gal}(M)$ is the subgroup of $g \in \mathrm{GIso}(V, \psi)$ for which there exists $\tau = \tau(g) \in G_K$ such that the actions of g and τ on $(V^{\otimes n})_{p,p}$ coincide for all n. By construction, we have an exact sequence

$$1 \to \mathrm{Gal}(M)^\circ = \mathrm{MT}(M) \to \mathrm{Gal}(M) \to \underline{\mathrm{Gal}_{L/K}} \to 1$$

of algebraic groups over $\mathbb{Q}$, where L is some finite extension of K. (Here and throughout, G° denotes the maximal connected subgroup of G.)

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The Sato-Tate group

Define the *algebraic Sato-Tate group*

$$\mathrm{AST}(M) = \mathrm{Gal}(M) \cap \mathrm{GIso}(V, \psi)^\circ;$$

note that $\mathrm{GIso}(V, \psi)^\circ$ equals $\mathrm{Sp}(V, \psi)$ or $\mathrm{SO}(V, \psi)$.

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The *Sato-Tate group* $\mathrm{ST}(A)$ is a maximal compact subgroup of $\mathrm{AST}(M)_{\mathbb{C}}$. We have an exact sequence of compact Lie groups

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Endomorphisms and Sato-Tate groups

Put $M = H^1(A)$ for A/K an abelian variety of dimension $g > 0$. Then

$$\mathrm{GIso}(V, \psi) \cong \mathrm{GSp}(2g) \qquad \text{and} \qquad (V^{\otimes 2})^{1,1} \cong \mathrm{End}(A_{\overline{K}})_{\mathbb{Q}}.$$

In many cases (e.g., when $g \leq 3$), the map

$$((V^{\otimes 2})^{1,1})^{\otimes n} \to (V^{\otimes 2n})^{n,n}$$

is surjective, so $\mathrm{AST}(M)$ and $\mathrm{ST}(M)$ are determined entirely by endomorphisms. In these cases, the exact sequence

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Warmup: elliptic curves

If $A = E$ is of dimension $g = 1$, then

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- If E has no CM, then $\mathrm{AST}(M) = \mathrm{SL}(2)$ and $\mathrm{ST}(M) = \mathrm{SU}(2)$.
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- If E has CM in an overfield L/K, then $\mathrm{ST}(M)$ is the normalizer of $\mathrm{ST}(M_L) = \mathrm{SO}(2, \mathbb{R})$ in $\mathrm{SU}(2)$.

This illustrates a general phenomenon: for fixed parameters, there are generally infinitely many options for the $\mathbb{Q}$-algebraic group $\mathrm{AST}(M)$. By contrast, $\mathrm{ST}(M)$ depends only on $\mathrm{AST}(M)_{\mathbb{R}}$, for which there are only finitely many options.

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This illustrates a general phenomenon: for fixed parameters, there are generally infinitely many options for the $\mathbb{Q}$-algebraic group $\mathrm{AST}(M)$. By contrast, $\mathrm{ST}(M)$ depends only on $\mathrm{AST}(M)_{\mathbb{R}}$, for which there are only finitely many options.

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If $A = E$ is of dimension $g = 1$, then

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Properties of Sato-Tate groups

For M as above, the group $\mathrm{ST}(M)$ satisfies the following conditions.

- (ST1) The group $\mathrm{ST}(M)$ is a closed subgroup of $\mathrm{USp}(2g)$. (Equality is the generic case.)
- (ST2) The connected group $\mathrm{ST}(M)^\circ$ is the closure of the subgroup generated by **Hodge circles**: images of cocharacters $\theta : \mathrm{U}(1) \to \mathrm{ST}(M)^\circ$ with weight $p - q$ of multiplicity $h^{p,q}$.
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Theorem (well-known)

For $g = 2$, there are exactly 6 conjugacy classes of subgroups of $\mathrm{USp}(4)$ *which can occur as* $\mathrm{ST}(A)^\circ$, *isomorphic to*

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This list corresponds to the possibilities for $\mathrm{End}(A_{\overline{K}})_{\mathbb{R}}$:

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Consequently, the passage from A to $\mathrm{ST}(A)^\circ$ conflates distinct geometric behaviors. For instance, a simple CM abelian fourfold gives the same group $\mathrm{U}(1) \times \mathrm{U}(1)$ as the product of two nonisogenous CM elliptic curves, as in both cases $\mathrm{End}(A_{\overline{K}})_{\mathbb{R}} \cong \mathbb{C} \times \mathbb{C}$.

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Theorem ([FKRS])

Take $g = 2$.

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(b) *Of these, exactly* 52 *are realized as* $\mathrm{ST}(M)$ *for suitable* A. *The generic case* $\mathrm{ST}(M) = \mathrm{USp}(4)$ *occurs iff* $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$.

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Consequences for abelian surfaces

For $g = 2$, we read off some arithmetic consequences.

Corollary (improvement of a result of Silverberg)

The minimal field L/K with $\mathrm{End}(A_L) = \mathrm{End}(A_{\overline{K}})$ has degree dividing 48. This bound is realized even for $K = \mathbb{Q}$, e.g., by the Jacobian of $y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$.

Corollary

The density of prime ideals with zero Frobenius trace exists and belongs to

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All of these cases are realized, e.g, 7/8 by $y^2 = x^5 + 2x$. (Only the case 3/8 cannot occur for $K = \mathbb{Q}$.)

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By contrast, suppose that M is *discrete* in the sense of Gross's lecture, i.e., the centralizer of $\mathrm{ST}(M)^\circ$ in $\mathrm{USp}(2g)$ is finite. Then one gets a finite list of options *even without (ST3)*. One only needs to describe the subgroups of the group $\mathrm{Out}(\mathrm{ST}(M)^\circ)$; that group consists (approximately) of automorphisms of the Dynkin diagram.

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Recall that to compute $\mathrm{ST}(M)$, we have to look at $(V^{\otimes n})^{p,p}$ whenever $n > 0$, nw is even, and $p = nw/2$. For $n = 1$, this is $\mathrm{NS}(X_{\overline{K}})_{\mathbb{Q}}$ by the Lefschetz $(1,1)$ theorem.

Put

$$\rho = \operatorname{rank} \mathrm{NS}(X), \qquad \overline{\rho} = \operatorname{rank} \mathrm{NS}(X_{\overline{K}}).$$

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(b) *If E is totally real, then $\mathrm{AST}(M)^\circ = \mathrm{Res}^E_{\mathbb{Q}} \mathrm{SO}(V_{\mathrm{tr}}, \psi)$.*

(c) *If E is CM, then $\mathrm{AST}(M)^\circ = \mathrm{Res}^E_{\mathbb{Q}} \mathrm{U}(V_{\mathrm{tr}}, \psi)$.*

Aside: the Mumford-Tate conjecture holds for X (Tankeev, 1995).

Zarhin's theorem for Kummer surfaces

For X the Kummer of an abelian surface A, we have

$$\mathrm{ST}(H^2(X))^\circ = \mathrm{ST}(H^1(A))^\circ/\{\pm 1\}, \quad \mathrm{ST}(H^2(X)) = \mathrm{ST}(H^1(A))/\{\pm 1\}.$$

How does this relate to Zarhin's theorem?

$\mathrm{ST}(H^1(A))^\circ$	$\mathrm{ST}(H^2(X))^\circ$	$\overline{\rho}$	$E_{\mathbb{R}}$
U(1)	U(1)	20	$\mathbb{C}$
SU(2)	SO(3)	19	$\mathbb{R}$
U(1) × U(1)	U(1) × U(1)	18	$\mathbb{C} \times \mathbb{C}$
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Exercise (open!)

Recover the classification of Sato-Tate groups of abelian surfaces. Do non-Kummer surfaces with $\overline{\rho} = 18$ account for the 3 missing groups?

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A classification of Sato-Tate groups?

Problem

Using Zarhin's theorem, classify the possible Sato-Tate groups associated to K3 surfaces of arbitrary rank. In particular, what are the possible zero trace densities besides $0, 1/2$*?*

Note that given $\overline{\rho}$ and $E_{\mathbb{R}}$, $\mathrm{ST}(M)$ is determined by its action on $\mathrm{NS}(X_{\overline{K}})_{\mathbb{R}}$ (because $\mathrm{ST}(M)^{\circ}$ is "as large as possible").

Beware that unlike for abelian varieties, one is unlikely to find interesting examples "by accident" (compare Jahnel's talk). We'll discuss the reason later.

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Theorem ([FKS])

Take M as above.

(a) *There are 26 conjugacy classes of subgroups of* $\mathrm{USp}(4)$ *satisfying (ST1), (ST2), (ST3).*

(b) *Of these, at least 25 are realized as* $\mathrm{ST}(M)$ *for suitable M.*

Due to the changed position of the Hodge circles, the options for $\mathrm{ST}(M)^\circ$ are not the same as for abelian surfaces:

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The maximum component order is 12. The zero densities are $0, 1/2, 3/4$ and possibly $5/8$.

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By contrast, for motives of weight greater than 1, a Hodge structure cannot vary arbitrarily in families; its variation is constrained by Griffiths transversality (thus precluding a universal family). Refining a prediction of de Jong, the generalized André-Oort conjecture (see Klingler's AMS SLC 2015 lecture) suggests that jumping can only occur on a Zariski dense subset if the family "arises from a Shimura variety."

This is consistent with our experimental data: in the Dwork pencil, one expects that over all K, only finitely many fibers have $\mathrm{ST}(M) \neq \mathrm{USp}(4)$. Over $\mathbb{Q}$, we found *no* such examples (excluding the Fermat fiber).

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