Zeta functions of K3 surfaces over finite fields

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These slides can be downloaded from https://kskedlaya.org/slides/.

Arithmetic, Geometry and Computations on K3 Surfaces
ICERM, Providence, RI
September 17, 2026

Supported by NSF (grant DMS-2401536) and UC San Diego (Warschawski Professorship).

I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

Contents

Hasse–Weil zeta functions

For X a scheme of finite type over $\mathbb{Z}$, the **zeta function** is the complex analytic function defined for $\mathrm{Real}(s) \gg 0$ by the Euler product

$$\zeta(X, s) := \prod_{x \in X^\circ} (1 - \#\kappa(x)^{-s})^{-1}$$

where X° runs over closed points of the scheme X and $\kappa(x)$ is the residue field of x.

For example, for $X = \operatorname{Spec} \mathbb{Z}$ this gives the Riemann zeta function. More generally, for K a number field and $\mathfrak{o}_K$ the ring of integers, taking $X = \operatorname{Spec} \mathfrak{o}_K$ gives the Dedekind zeta function of K.

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Zeta functions of varieties over a finite field

Hereafter, fix a prime power $q = p^f$ and let $\mathbb{F}_q$ be the finite field of order q. For X an algebraic variety (or any scheme of finite type) over $\mathbb{F}_q$, we have

$$\zeta(X, s) = Z(X, q^{-s})$$

for the power series $Z(X, T)$ over $\mathbb{Z}$ given by

$$Z(X, T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right).$$

This power series always represents a **rational function**. For example, for X a genus 1 curve,

$$Z(X, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)} \text{ where } a := q + 1 - \#X(\mathbb{F}_q) \text{ satisfies } |a| \leq 2\sqrt{q} \text{ (Hasse).}$$

Equivalently, the numerator factors over $\mathbb{C}$ as $(1 - \alpha T)(1 - \beta T)$ with $|\alpha| = |\beta| = q^{1/2}$.

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Zeta functions of abelian varieties

For X an abelian variety of dimension g over $\mathbb{F}_q$, Weil showed that[1]

$$Z(X,T) = \prod_{i=0}^{2g} (\wedge^i L_X)(T)^{(-1)^{i+1}} = \frac{(\wedge^1 L_X)(T)(\wedge^3 L_X)(T)\cdots}{(\wedge^0 L_X)(T)(\wedge^2 L_X)(T)\cdots}$$

for a certain polynomial $L_X(T)$ of degree $2g$ which satisfies

$$L_X(0) = 1, \qquad q^g L_X(q^{-1}T) = T^{2g} L_X(qT^{-1})$$

and the reciprocal roots $\alpha_1, \dots, \alpha_{2g}$ in $\mathbb{C}$ satisfy $|\alpha_i| = q^{1/2}$. More concretely,

$$\#X(\mathbb{F}_{q^n}) = \prod_{i=1}^{2g} (1 - \alpha_i^n) \qquad (n = 1, 2, \dots).$$

[1] Notation gloss: If $F\colon V \to V$ satisfies $\det(1 - FT, V) = L_X(T)$, then $\det(1 - FT, \wedge^i V) = (\wedge^i L_X)(T)$.

The inverse problem for abelian varieties

By Tate's theorem, the zeta function is a **complete isogeny invariant** for abelian varieties over $\mathbb{F}_q$. One can refine this to get a complete **isomorphism** invariant; more on this later.

Moreover, it is known exactly which polynomials $L_X(T)$ can occur in the zeta function of some abelian variety X (Honda–Tate, Milne–Waterhouse): the obstruction cannot occur for $q = p$, is rare for $q \neq p$, and is (easily) explicitly computable. Using this, one can tabulate isogeny classes of abelian varieties; see LMFDB.

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By a **K3 surface** over a field k of any characteristic, I mean a smooth, projective, geometrically irreducible surface X which is simply connected[2] with trivial canonical bundle.

This talk will address two general questions about K3 surfaces (and related varieties).

- What can we say about zeta functions of K3 surfaces over finite fields?
- What does this tell us about the geometry and arithmetic of K3 surfaces?

Importantly, neither of these questions is entirely concentrated in the setting of working over a finite field! On one hand, some of the information we get about zeta functions over finite fields comes from working over $\overline{\mathbb{F}}_q$ or even lifting to characteristic 0. On the other hand, the information we get is also relevant in characteristic 0 (cf. Edgar's minicourse).

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Motivation: Kuga–Satake

Over $\mathbb{C}$, the Kuga–Satake construction presents the Hodge structure of a K3 surface using the tensor square of the Hodge structure of some (large) abelian variety. Using the language of **Shimura varieties**, this construction can be described in a way that makes sense over finite fields, even[3] in characteristic 2.

For example, this relationship was used by Deligne to establish the properties of zeta functions of K3 surfaces **before** he had proved the Weil conjectures in general. See next slide for a precise statement.

It is also used to prove the Tate conjecture for $H^2(X)$ and $H^2(X \times_{\mathbb{F}_q} X)$ (Ito–Ito–Koshikawa + much earlier work). We will (silently) use many consequences of this later.

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For X a K3 surface over $\mathbb{F}_q$,

$$Z(X,T) = (1-T)^{-1}(1-q^2T)^{-1}qQ(qT)^{-1}$$

where the polynomial $Q(T)$ satisfies $Q(T) \in q + T\mathbb{Z}[T]$, $\deg(Q) = 22$, $Q(1) = 0$, and

$$Q(T) = \prod_{i=1}^{22}(1-\alpha_i T), \qquad a_i \in \mathbb{C}, \qquad |\alpha_i| = 1.$$

The polynomial $Q(T)$ is reciprocal up to sign:

$$T^{22}Q(T^{-1}) = (-1)^{\mathrm{ord}_{T=-1}(Q)}Q(T).$$

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Factoring the zeta function

There is a unique factorization of polynomials $Q(T) = Q_{\mathrm{alg}}(T) Q_{\mathrm{trans}}(T)$ where

- $Q_{\mathrm{alg}}(0) = 1$ and **every** root of Q_{alg} is a root of unity;
- $Q_{\mathrm{trans}}(0) = 1$ and **no** root of Q_{trans} is a root of unity.

The degree of $Q_{\mathrm{alg}}(T)$ is the **geometric** Picard rank $\overline{\rho} = \operatorname{rank} \operatorname{Pic}(X_{\overline{\mathbb{F}}_q})$, which is always even. By contrast, the Picard rank $\rho = \operatorname{rank} \operatorname{Pic}(X)$ equals $\operatorname{ord}_{T=1}(Q_{\mathrm{alg}}) = \operatorname{ord}_{T=1}(Q)$ which has no parity constraint.

Note that $\overline{\rho} = 22$ is possible even though this cannot occur over $\mathbb{C}$! In this case we say X is **supersingular** (e.g., the Fermat quartic when $q \equiv 3 \pmod 4$). However, Artin showed that $\rho \neq 22$ if $q = p^f$ with f odd.

Question

For X supersingular, are there any other constraints on Q_{alg} in terms of q? Special case: for every $m \in \{-20, \dots, 22\}$, can we have $\#X(\mathbb{F}_q) = q^2 + mq + 1$ for suitable q, X?

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Lower bounds on point counts

Let X be a K3 surface over $\mathbb{F}_q$. By Deligne's theorem and the triangle inequality

$$\#X(\mathbb{F}_q) = q^2 + 1 + q\sum_{i=1}^{22} \alpha_i \geq q^2 + q + 1 - 21q = q^2 - 20q + 1.$$

In particular, if $q \geq 23$ then $X(\mathbb{F}_q) \neq \emptyset$.

Question

For which q does there exist a K3 surface X over $\mathbb{F}_q$ with $X(\mathbb{F}_q) = \emptyset$?

Such examples are known for $q = 2, 3, 4, 5$. One example for $q = 5$ is the Fermat quartic.

Theorem (Bernasconi–Filipazzi, 2025)

Let $q = p^f$ be a prime power with $p > 5$ for which there exists no K3 surface X over $\mathbb{F}_q$ with $X(\mathbb{F}_q) = \emptyset$ (e.g., $q = p \geq 23$). Then every smooth, projective, geometrically integral 3-fold over $\mathbb{F}_q$ with nef anticanonical class and negative Kodaira dimension has a rational point.

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Positivity constraints

Turning this around, the condition $\#X(\mathbb{F}_q) \geq 0$ imposes a nontrivial constraint on the zeta functions for $q \leq 19$. Some other nontrivial constraints of this type are the following.

- $\#X(\mathbb{F}_{q^2}) \geq \#X(\mathbb{F}_q)$ (for $q = 2, 3, 4$)
- $\#X(\mathbb{F}_{q^3}) \geq \#X(\mathbb{F}_q)$ (for $q = 2$).
- $\#X(\mathbb{F}_{q^4}) \geq \#X(\mathbb{F}_{q^2})$ (for $q = 2$).

Aside: for K3 surfaces, the inverse problem for zeta functions is solved "up to base extension" if $p > 5$ (Taelman, Ito). Any attempt to eliminate the base extension would have to account for these positivity constraints.[4]

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Turning this around, the condition $\#X(\mathbb{F}_q) \geq 0$ imposes a nontrivial constraint on the zeta functions for $q \leq 19$. Some other nontrivial constraints of this type are the following.

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A sharper Weil bound for Q_{trans}

One obtains better bounds on the contribution of Q_{trans} to $\#X(\mathbb{F}_{q^n})$ using the fact that for every cyclotomic polynomial Φ_i (of degree $\varphi(i)$),

$$\varphi(i) \log q + \sum_{\alpha:\ Q_{\text{trans}}(\alpha)=0} \log |\Phi_i(\alpha)| = \log |\text{Res}(Q_{\text{trans}}, \Phi_i)| \geq 0$$

plus a little bit of linear programming (and some exhaustive searches).

Theorem (K, 2026+)

Let X be a K3 surface over $\mathbb{F}_q$ with $X(\mathbb{F}_q) = \emptyset$. If $q = 7, 8$, then $\overline{\rho} \geq 4$; if $q = 9$, then $\overline{\rho} \geq 6$; if $q = 11$, then $\overline{\rho} \geq 8$; if $q = 13$, then $\overline{\rho} \geq 10$, if $q = 16, 17$, then $\overline{\rho} \geq 16$; if $q = 19$, then $\overline{\rho} \geq 18$.

This means that one can potentially rule out the larger primes by analyzing a small number of options for $Q(T)$. (Note: if $\overline{\rho} = 22$ then $\#X(\mathbb{F}_q) \equiv 1 \pmod{q}$.)

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… and related varieties

From the zeta function $Z(X, T)$, one can compute the zeta functions of Hilbert schemes associated to X. This gives rise to additional positivity considerations.

Theorem (Debarre–Laface–Roulleau $+\epsilon$)

For $q \neq 3$, any smooth cubic fourfold over $\mathbb{F}_q$ contains an $\mathbb{F}_q$-rational line.

For $q \geq 5$, expressing the point count on the Fano variety of lines $F(X)$ in terms of $Z(X, T)$ gives a positive lower bound. For $q = 2, 4$, one instead shows that

$$\#F(X)(\mathbb{F}_q) \equiv 1 + m + m^2 \pmod{q}, \qquad m := \mathrm{Trace}(\mathrm{Frob}_q, H^2(F(X), \mathbb{Q}_\ell)) \in \mathbb{Z}$$

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Since the only missing case is $q = 3$, and the moduli space of cubic fourfolds has roughly $3^{19} \approx 10^9$ points over $\mathbb{F}_3$, it is feasible to fill this gap using a census.

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Computing zeta functions

In principle, one can compute the zeta function of a K3 surface X over $\mathbb{F}_q$ by enumerating $X(\mathbb{F}_{q^n})$ for $n = 1, \dots, 10$ plus the sign of the functional equation.[5] In practice this is only feasible for $q = 2$ or in cases with a big Picard sublattice (with known Galois action).

Instead, one can interpret $Q(T)$ as the (reversed) charpoly of Frobenius on H^2 for a suitable cohomology theory. The most practical for computation is **not** étale (ℓ-adic) cohomology but rather crystalline (p-adic) cohomology. This is a long story with contributions from many people (K, Tuitman, Harvey, Costa, etc.).

Executive summary: if you choose a smooth lift of X over the Witt vectors $W(\mathbb{F}_q)$, then the algebraic de Rham cohomology admits a Frobenius action given by a certain p-adically convergent power series. Expressing this in terms of generators of cohomology involves a lot of pole order reduction (e.g., Griffiths–Dwork for a projective hypersurface).

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However, one would like this also for members of the family with ADE singularities. In particular this would add many examples with elevated geometric Picard rank, and in particular many more supersingular examples (perhaps with new values of Q_{alg}).

A related task is making censuses of curves of genus g over $\mathbb{F}_q$ (Huang–K–Lau). Recently Bergström has reported some success using AI to accelerate these censuses.

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The absolute Galois group $G_{\mathbb{F}_q}$ acts on $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})$ preserving the pairing and the ample cone. For $\mathrm{Frob}_q \in G_{\mathbb{F}_q}$ the canonical generator, we have

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The crystalline Torelli theorem was proved by Ogus for q odd:[7] an isomorphism $X \to Y$ of supersingular K3 surfaces over $\mathbb{F}_q$ is specified by an isomorphism $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q}) \to \mathrm{Pic}(Y_{\overline{\mathbb{F}}_q})$ preserving all of

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The group $\# \mathrm{Br}(X)$ carries an alternating pairing, so its order is a square.

The same formula applies after base extension. Using this we can write down candidates for the lattice $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})$ and its Galois action that are consistent with Artin–Tate.

Remark of Elsenhans–Jahnel: sometimes there are no candidates! When $\mathrm{ord}_{T=-1}(Q) = 0$, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbb{F}_{q^2}})$ is an isomorphism and so $\#A_{\mathrm{Pic}(X)} = \#A_{\mathrm{Pic}(X_{\mathbb{F}_{q^2}})}$; since $\mathrm{ord}_{T=1}(Q) + \mathrm{ord}_{T=-1}(Q)$ is always even, this implies that $Q(-1)$ must be a square.

[8] If X is supersingular with Artin invariant σ, the right-hand side should be divided by $q^{\dim \underline{\mathrm{Pic}}(X)} = q^{\dim \underline{\mathrm{Pic}}_0(X)}$. The group scheme $\underline{\mathrm{Pic}}_0$ can have dimension as large as 2σ even though its group of $\overline{\mathbb{F}}_q$-points is trivial.

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The same formula applies after base extension. Using this we can write down candidates for the lattice $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})$ and its Galois action that are consistent with Artin–Tate.

Remark of Elsenhans–Jahnel: sometimes there are no candidates! When $\mathrm{ord}_{T=-1}(Q) = 0$, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{\mathbb{F}_{q^2}})$ is an isomorphism and so $\# A_{\mathrm{Pic}(X)} = \# A_{\mathrm{Pic}(X_{\mathbb{F}_{q^2}})}$; since $\mathrm{ord}_{T=1}(Q) + \mathrm{ord}_{T=-1}(Q)$ is always even, this implies that $Q(-1)$ must be a square.

[8] If X is supersingular with Artin invariant σ, the right-hand side should be divided by $q^{\dim \underline{\mathrm{Pic}}(X)} = q^{\dim \underline{\mathrm{Pic}}_0(X)}$. The group scheme $\underline{\mathrm{Pic}}_0$ can have dimension as large as 2σ even though its group of $\overline{\mathbb{F}}_q$-points is trivial.

Artin–Tate formula

If X is not supersingular[8], then

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Lifts to characteristic zero

Now suppose that X is not supersingular, so $\overline{\rho} \leq 20$. In this case, there always exists a lift $\mathfrak{X}$ of X over some finite totally ramified extension $\mathfrak{o}$ of $W(\mathbb{F}_q)$ such that the reduction map

$$\mathrm{Pic}(\mathfrak{X}_{\mathbb{C}}) \cong \mathrm{Pic}(\mathfrak{X}) \to \mathrm{Pic}(X_{\overline{\mathbb{F}}_q})$$

is an isomorphism (normally it is only injective).

This means that just as over $\mathbb{C}$, $\mathrm{Pic}(X_{\overline{\mathbb{F}}_q})$ (which has signature $(1, \overline{\rho} - 1)$) admits a primitive embedding into the K3 lattice $E_8^{\oplus 2} \oplus U^{\oplus 3}$. There is no way to construct this directly from X because there is no meaningful interpretation of $H^2(X_{\overline{\mathbb{F}}_q}, \mathbb{Z})$.

For $p > 2$, Inoue[9] established the existence of a **quasi-canonical lift** of X. In particular the transcendental Hodge structure of the lift has CM in the sense of Zarhin.

[9] For $p > 3$ this was done by Nygaard–Ogus. For $p = 2$, Ito–Ito–Koshikawa produce a quasi-canonical lift after a base extension.

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Dare to dream: Deligne modules?

One can use CM lifts to give "linear algebraic" descriptions of the category of abelian varieties over $\mathbb{F}_q$ (Deligne, Centeleghe–Stix, Bergström–Karemaker–Marseglia, Rybakov). These are practical for tabulating **isomorphism classes** of abelian varieties (Marseglia), as opposed to **isogeny classes** which currently occur in LMFDB.

Concrete example: define a **Deligne module** to be a finite free abelian group with two operators F, V such that $FV = VF = q$ and $F \otimes \mathbb{Q}$ is semisimple. Deligne showed that the category of **ordinary** abelian varieties over $\mathbb{F}_q$ is equivalent to the category of Deligne modules for which the characteristic polynomial of $F \otimes \mathbb{Q}$ is an ordinary q-Weil polynomial.

Question

Is there a corresponding description of the category of K3 surfaces over $\mathbb{F}_q$?

This would potentially help us answer many of the questions raised earlier...

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That QR code again