$L$-functions via deformations:
from hyperelliptic curves to hypergeometric motives

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Arithmetic of Hyperelliptic Curves
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Zeta functions of algebraic varieties

For $X$ an algebraic variety over a finite field $\mathbb{F}_q$, the zeta function

$$\zeta(X, T) = \prod_{x \in X^0} (1 - T^{[\kappa(x):\mathbb{F}_q]})^{-1} = \exp \left( \sum_{n=1}^\infty \frac{\#X(\mathbb{F}_{q^n})}{n} T^n \right) \in \mathbb{Z}[T]$$

is a rational function of $T$. That is because it is possible to a spectral interpretation of $\zeta(X, T)$ consisting of a field $K$ of characteristic 0; finite-dimensional $K$-vector spaces $V_i$ for $i = 0, 1, \ldots, 2 \dim(X)$; and $K$-linear endomorphisms $F_i$ on $V_i$ satisfying the Lefschetz trace formula:

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2 \dim(X)} (-1)^i \, \text{trace}(F_i^n, V_i) \quad (n = 1, 2, \ldots).$$

This then implies that

$$\zeta(X, T) = \prod_{i=0}^{2 \dim(X)} \det(1 - F_i T, V_i)^{(-1)^i+1}.$$
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Weil cohomology: $\ell$-adic versus $p$-adic

Such data are provided in a systematic way by *Weil cohomology* constructions, of which there are two general types.

- Grothendieck’s formalism of *étale cohomology* produces one Weil cohomology theory with coefficients in $\mathbb{Q}_\ell$ for each prime $\ell$ other than $p$, the characteristic of $\mathbb{F}_q$. This theory is quite rich, and has been the basis for most new developments on geometric zeta functions.

- Building on Dwork’s original proof of rationality (predating *étale cohomology!*), Berthelot introduced *rigid cohomology* with coefficients in a finite extension\(^1\) of $\mathbb{Q}_p$. (This relates explicitly to *crystalline cohomology* for smooth proper varieties or *Monsky-Washnitzer cohomology* for smooth affine varieties.) Recently, most formalism of *étale* cohomology has been replicated for rigid cohomology.

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K.S. Kedlaya

L-functions via deformations

Trieste, August 24, 2017

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Suppose $X$ is smooth proper over $\mathbb{F}_q$. By Deligne’s analogue of the Riemann hypothesis, there exists a unique factorization

$$\zeta(X, T) = \prod_{i=0}^{2 \dim(X)} P_i(T)(-1)^{i+1}$$

in which $P_i(T) \in 1 + T \mathbb{Z}[T]$ has all $\mathbb{C}$-roots of absolute value $q^{-i/2}$.

More precisely, Deligne (1974, 1980) showed that for the data $F_i, V_i$ arising from $\ell$-adic étale cohomology, the polynomial $P_i(T) = \det(1 - F_i T, V_i)$ has all $\mathbb{C}$-roots of absolute value $q^{-i/2}$. A variant of the second proof can be executed with rigid cohomology (K, 2006).

There is a formal process for “factoring” $X$ into pieces that account for the individual $P_i$; this is the theory of motives.
Factorization of zeta functions and varieties

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Zeta functions and $L$-functions

Suppose now that $X$ is a smooth proper variety over a number field $K$. Then for the motive of weight $i$ associated to $X$, one gets an $L$-function by taking an Euler product $\prod_p L_p(s)$ in which for almost all prime ideals $p$ of $\mathfrak{O}_K$, we have $L_p(s) = P_i(\text{Norm}(p)^{-s})$ where $P_i$ is the corresponding factor of the zeta function of the reduction of (an integral model of) $X$ modulo $p$.

This may be familiar for $X = E$ an elliptic curve. Over $\mathbb{F}_q$, we have
\[
P_0(T) = 1 - T, \quad P_1(T) = 1 - a_E T + qT^2, \quad P_2(T) = 1 - qT.
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Over $K$, for $i = 0, 1, 2$, the resulting $L$-functions are
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\zeta_K(s), \quad L(E, s), \quad \zeta_K(s - 1)
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where $L(E, s)$ is (almost) $\prod_p (1 - a_{E,p} q^{-s} + q^{1-2s})^{-1}$ for $q = \text{Norm}(p)$. Similar considerations apply when $X$ is a hyperelliptic (or arbitrary) curve.
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Computational aspects of Weil cohomology

With a few exceptions\(^2\), the only methods we know for computing \(\zeta(X, T)\) are to explicitly compute the matrices via which \(F_i\) act on some basis of \(V_i\), for some choice of Weil cohomology.

The definition of étale cohomology, which quantifies over all covers in the étale topology, is hard to make computationally effective. This can be done for curves of low genus (using the Jacobian as in Schoof’s method) and for motives attached to modular forms (Edixhoven et al.).

By contrast, rigid cohomology can be defined\(^3\) more concretely in terms of differential forms on certain \(p\)-adic rigid analytic spaces. Correspondingly, it tends to be a better source for algorithms.

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The roots of $P_1(T)$ in $\mathbb{C}$ lie on the circle $|T| = q^{-1/2}$ (Weil). Aside: the class group of $X$ (a/k/a $\#J(\mathbb{F}_q)$ for $J$ the Jacobian of $X$) has order $P_1(1)$.

As per the general setup, we wish to compute $P_1(T)$ as $\det(1 - FT, V)$ for suitable $F$ acting on suitable $V$.

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Suppose $p \neq 2$ and $X$ is a hyperelliptic curve of genus $g$ with a rational Weierstrass point, which then admits an affine model $y^2 = Q(x)$ with $Q$ monic of degree $2g + 1$. We may then take $V$ to be the first (algebraic) de Rham cohomology of a smooth lift of $X$ over the unramified extension $K$ of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$.

Concretely, for $\tilde{Q}$ a monic lift of $Q$, we have $V = \bigoplus_{i=0}^{2g-1} K \cdot \frac{x^i dx}{2y}$, with a quite explicit recipe for rewriting general differentials in terms of these.

The action of $F$ is given by $x \mapsto x^q$ and (as a series)

$$y \mapsto y^q \left(1 + \frac{\tilde{Q}(x^q) - \tilde{Q}(x)^q}{\tilde{Q}(x)^q} \right)^{1/2}.$$ 

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Let $X_t$ be a family of hyperelliptic curves over $\mathbb{F}_q$ in one parameter $t$, lifted to a family $\tilde{X}_t$ over $K$. Then the relative de Rham cohomology of $\tilde{X}_t$ over the $t$-line forms a vector bundle of rank $2g$ away from the bad fibers, with the added structure of a Gauss-Manin connection.

Moreover, over a certain rigid-analytic subspace of the $t$-line, this connection admits a Frobenius structure which specializes to the Frobenius matrices described on the previous slide.

This gives an alternate approach to computing zeta functions, which is implemented in Magma (Hubrechts, Tuitman). There is also an implementation by Sebastian Pancratz as an optional Sage package.

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In the context of the previous slide, there exist a $2g \times 2g$ matrix $N$ over $K(t)$ with poles at the bad fibers\footnote{and possibly more points depending on the choice of a basis of the vector bundle.} and a $2g \times 2g$ matrix $F$ whose entries are rigid analytic functions defined away from some neighborhoods of the poles of $N$, satisfying the commutation relation

$$NF - pF \sigma(N) + t \frac{dF}{dt} = 0$$

where $\sigma$ is the substitution $t \mapsto t^p$.

For any $\lambda \in \mathbb{F}_q$, let $[\lambda] \in K$ be its Teichmüller lift. Then $F([\lambda])$ equals the Frobenius matrix acting on some basis of the rigid cohomology of $X_{\lambda}$.

The matrix $N$ can typically be computed easily. This imposes a differential equation on the entries of $F$, which can be solved after establishing an initial condition, e.g., by running the direct method on one fiber.
The deformation method: concrete interpretation

In the context of the previous slide, there exist a $2g \times 2g$ matrix $N$ over $K(t)$ with poles at the bad fibers\(^5\) and a $2g \times 2g$ matrix $F$ whose entries are rigid analytic functions defined away from some neighborhoods of the poles of $N$, satisfying the commutation relation

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Why are HGMs better examples than HECs?

To illustrate the deformation method, we will use hypergeometric motives (HGMs) instead of hyperelliptic curves (HECs). Why?

- For HECs, one must compute the Gauss-Manin connection for a suitable one-parameter family. For HGMs, this is replaced with a simple explicit formula.
- For HECs, the connection may have many singularities, which contribute to the complexity of subsequent calculations. For HGMs, the only bad points are $t = 0, 1, \infty$.
- For HECs, we must develop\(^6\) power series solutions at some point. For HGM, these are given explicitly by hypergeometric series.
- For HECs, we need an outside source for the initial condition on Frobenius. For HGMs, (conjecturally) there is an explicit formula.

\(^6\)Silver lining: there is a quadratically convergent method for this.
Why are HGMs better examples than HECs?

To illustrate the deformation method, we will use hypergeometric motives (HGMs) instead of hyperelliptic curves (HECs). Why?

- For HECs, one must compute the Gauss-Manin connection for a suitable one-parameter family. For HGMs, this is replaced with a simple explicit formula.
- For HECs, the connection may have many singularities, which contribute to the complexity of subsequent calculations. For HGMs, the only bad points are $t = 0, 1, \infty$.
- For HECs, we must develop power series solutions at some point. For HGM, these are given explicitly by hypergeometric series.
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The hypergeometric trace formula

Magma’s HGM package computes Euler factors of the associated $L$-functions using a trace formula derived from Greene’s finite hypergeometric functions. The trace over $\mathbb{F}_q$ for the parameter $t$ equals

$$\frac{1}{1-q} \sum_{r=0}^{q-2} \omega_p(M/t)^r Q_q(r)$$

where $\omega_p$ is the Teichmüller character and

$$Q_q(r) = (-1)^{m_0} q^{D+m_0-mr} G_q(r)$$

where $G_q(r) = \prod_v g_q(rv)^{\gamma_v}$ where

$$g_q(a) = \sum_{u \in \mathbb{F}_q^\times} \omega_p(u)^{-a} \zeta_p^{\text{trace}_{\mathbb{F}_q/\mathbb{F}_p}(u)}$$

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Traces vs. deformations

Following GP/Pari (Cohen), Magma (Watkins) computes the Gauss sum $g_v(r)$ very efficiently using the Gross-Koblitz formula

$$g_v(a) = -\pi S_p(a) \prod_{i=0}^{f-1} \Gamma_p \left( \frac{a(i)}{q-1} \right)$$

where $f = \log_p q$; $\pi$ is the $(p - 1)$-st root of $-p$ for which $\zeta_p \equiv 1 + \pi \pmod{\pi^2}$; $S_p(a)$ is the sum of the base-$p$ digits of $a$; $a^{(i)}$ is the remainder of $p^{-i}a$ modulo $q - 1$; and $\Gamma_p$ is the $p$-adic Gamma function.

This works well for computing $L$-functions: if you want all Fourier coefficients up to $X$, you only need traces for $q \leq X$.

However, if you want all the Euler factors for $p \leq X$ (e.g., to compute Sato-Tate statistics), you need traces for $q = p^1, \ldots, p^{\lfloor d/2 \rfloor}$ where $d$ is the degree of the Euler factor. By contrast, deformation computes the whole Frobenius matrix at once, so has complexity linear in $p$ rather than $q$. 
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Contents

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2 Hyperelliptic curves

3 Hypergeometric motives

4 A demonstration
The remainder of the lecture consists of an explicit calculation of HGM Euler factors using the deformation method. This demo is contained in a Jupyter notebook: click here.

Disclaimer: the correctness of these calculations depends on various missing facts. Some of these should be easy to obtain (e.g., the amount of working $p$-adic and $t$-adic precision required to obtain the final answers) and some may be more difficult (e.g., the formula for the initial condition of the Frobenius structure).

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