Relative p-adic Hodge theory and Rapoport-Zink period domains

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The Robba ring

Let K_0 be the completion of a (possibly infinite) unramified algebraic extension of $\mathbb{Q}_p$, equipped with the p-adic norm (with $|p| = p^{-1}$) and valuation (with $v_p(p) = 1$).

Let $\mathbf{B}^\dagger_{\mathrm{rig},K_0}$ be the *Robba ring* over K_0, i.e., the ring of series $\sum_{i \in \mathbb{Z}} c_i \pi^i$ with $c_i \in K_0$ which converge for π in some annulus of open outer radius 1. In other words, for each series, there exists $r > 0$ such that

$$\lim_{i \to \pm\infty} (v_p(c_i) + si) = +\infty \qquad (s \in (0, r]).$$

The Robba ring includes some elements with unbounded coefficients, e.g.,

$$t = \log(1 + \pi) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \pi^i.$$

Let $\mathbf{B}^\dagger_{K_0}$ be the subring of series with bounded coefficients; it is a henselian local field for the p-adic valuation. Its completion is denoted $\mathbf{B}_{K_0}$.

More structures on the Robba ring

Let φ be the unique p-power Frobenius lift on K_0. Equip $\mathbf{B}_{K_0}, \mathbf{B}^\dagger_{K_0}, \mathbf{B}^\dagger_{\mathrm{rig},K_0}$ with an action of the monoid $\mathbb{Z}_p - \{0\}$:

$$(p^j\gamma)\left(\sum_{i\in\mathbb{Z}} c_i\pi^i\right) = \varphi^j(c_i)((1+\pi)^{p^j\gamma} - 1)^i \qquad (j \geq 0, \gamma \in \mathbb{Z}_p^\times).$$

Identify $\mathbb{Z}_p - \{0\}$ with $\varphi^{\mathbb{Z}_{\geq 0}} \times \Gamma$ with φ corresponding to p and Γ to $\mathbb{Z}_p^\times$, so that φ is now extended to a Frobenius lift on $\mathbf{B}_{K_0}, \mathbf{B}^\dagger_{K_0}, \mathbf{B}^\dagger_{\mathrm{rig},K_0}$.

A field of norms construction

Choose a coherent sequence ϵ_n of primitive p^n-th roots of unity. One has a commuting diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\varphi} & \mathbf{B}^{\dagger}_{\mathrm{rig},K_0} & \xrightarrow{\varphi} & \mathbf{B}^{\dagger}_{\mathrm{rig},K_0} & \xrightarrow{\varphi} & \cdots \\
 & & \Big\downarrow{\scriptstyle \theta_n} & & \Big\downarrow{\scriptstyle \theta_{n+1}} & & \\
\cdots & \longrightarrow & K_0(\epsilon_n)[\![t]\!] & \longrightarrow & K_0(\epsilon_{n+1})[\![t]\!] & \longrightarrow & \cdots
\end{array}$$

in which θ_n is induced by completion at $\pi = \epsilon_n - 1$ (on the subring of $\mathbf{B}^{\dagger}_{\mathrm{rig},K_0}$ which converge on an annulus including this value), and the bottom horizontal arrows act as φ on K_0, fix ϵ_n, and carry t to pt.

Given a finite étale algebra S over $\mathbf{B}^{\dagger}_{\mathrm{rig},K_0}$, for n large we can form $S \otimes_{\theta_n} K_0(\epsilon)$. If we view the right side as a $K_0(\epsilon)$-algebra via φ^n, then this $K_0(\epsilon)$-algebra is independent of n; call it simply $S \otimes_{\theta} K_0(\epsilon)$.

Comparison of finite étale algebras

Let $\mathbf{A}_{K_0}, \mathbf{A}^\dagger_{K_0}$ denote the valuation subrings of $\mathbf{B}_{K_0}, \mathbf{B}^\dagger_{K_0}$.

Theorem

The categories of finite étale algebras over $\mathbf{A}_{K_0}, \mathbf{A}^\dagger_{K_0}$, and $K_0(\epsilon) = \cup_n K_0(\epsilon_n)$ are equivalent.

The equivalence from $\mathbf{A}_{K_0}$ to $\mathbf{A}^\dagger_{K_0}$ exists because $\mathbf{B}^\dagger_{K_0}$ is henselian. The equivalence from $\mathbf{A}^\dagger_{K_0}$ to $K_0(\epsilon)$ is given by

$$S \mapsto (S \otimes_{\mathbf{A}^\dagger_{K_0}} \mathbf{B}^\dagger_{\mathrm{rig},K_0}) \otimes_\theta K_0(\epsilon).$$

The inverse functor can be constructed directly, or using (φ, Γ)-modules.

(φ, Γ)-modules

Let K be a finite extension of K_0. By the previous theorem, we obtain finite extensions $\mathbf{B}_K, \mathbf{B}^\dagger_K, \mathbf{B}^\dagger_{\mathrm{rig},K}$ of $\mathbf{B}_{K_0}, \mathbf{B}^\dagger_{K_0}, \mathbf{B}^\dagger_{\mathrm{rig},K_0}$, to which the actions of φ and Γ extend.

A *φ-module* over one of $\mathbf{B}_K, \mathbf{B}^\dagger_K, \mathbf{B}^\dagger_{\mathrm{rig},K}$ is a finite free module over this ring, equipped with a semilinear action of φ whose action on some (hence any) basis is via an invertible matrix.

Let Γ_K be the open subgroup of Γ corresponding to $\mathrm{Gal}(K(\epsilon)/K)$ via the cyclotomic character. A *(φ, Γ_K)-module* is a φ-module further equipped with a continuous (...) action of Γ_K which commutes with φ.

Slope theory for φ-modules

Any unit in $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ belongs to $\mathbf{B}^{\dagger}_{K}$ (by considering Newton polygons), and so has a well-defined p-adic valuation. For D a φ-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, the *degree* of D is the valuation of the matrix via which φ acts on some (hence any) basis of D. The *slope* of D is $\mu(D) = \deg(D)/\operatorname{rank}(D)$.

A φ-module over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is *étale* if it admits a basis on which φ acts via an invertible matrix over the valuation subring $\mathbf{A}^{\dagger}_{K}$ of $\mathbf{B}^{\dagger}_{K}$.

Theorem (KSK, 2005)

A φ-module M over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is étale if and only if:

(a) $\mu(M) = 0$; *and*

(b) *for any φ-submodule M' of M, $\mu(M') \geq 0$.*

(φ, Γ)-modules and Galois representations

For K a finite extension of K_0, let G_K be the absolute Galois group of K.

Theorem (Fontaine, Cherbonnier-Colmez, Berger, KSK)

The categories of étale (φ, Γ_K)-modules over $\mathbf{B}_K, \mathbf{B}^\dagger_K, \mathbf{B}^\dagger_{\mathrm{rig},K}$ are equivalent to each other, and to the category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces.

To construct the representation from an étale (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, choose a basis on which φ acts via an invertible matrix over $\mathbf{A}^\dagger_K$. For each positive integer m, we can trivialize the action of φ modulo p^m over some connected finite étale extension of $\mathbf{A}^\dagger_K$. This extension carries an action of G_K; the action on φ-invariant elements modulo p^m gives a map $G_K \to \mathrm{GL}_d(\mathbb{Z}_p/p^m\mathbb{Z}_p)$. These are compatible as m varies, so we may take the inverse limit and then invert p.

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Filtered isocrystals

Let K be a complete (not necessarily finite) extension of K_0. A *filtered isocrystal* (or *filtered φ-module*) over K (relative to K_0) consists of

(a) a finite-dimensional K_0-vector space D equipped with an invertible semilinear action of φ; and

(b) an exhaustive decreasing filtration $\mathrm{Fil}^\bullet D_K$ on $D_K = D \otimes_{K_0} K$.

For example, if K is finite and totally ramified over K_0, and X is a smooth proper scheme over the valuation subring of K, then the de Rham cohomology of the generic fibre is a filtered isocrystal, using the Frobenius action on crystalline cohomology (transferred via the comparison isomorphism) and the Hodge filtration on de Rham cohomology.

Weak admissibility for filtered isocrystals

Let K be a complete extension of K_0. Let D be a filtered isocrystal over K.

As in the case of φ-modules, the determinant of the matrix of action of φ on a basis of D has valuation independent of the choice of basis. Call this valuation $t_N(D)$.

Define the *Hodge-Tate weights* of D as the multiset containing $i \in \mathbb{Z}$ with multiplicity $\dim_K \operatorname{Fil}^i D_K - \dim_K \operatorname{Fil}^{i+1} D_K$. Let $t_H(D)$ be the sum of the Hodge-Tate weights.

We say D is *weakly admissible* if

(a) $t_N(D) - t_H(D) = 0$; and

(b) for any φ-stable subspace D' of D, $t_N(D') - t_H(D') \geq 0$.

Weak admissibility and Galois representations

Let K be a *finite* extension of K_0. Let D be a filtered isocrystal over K, and put $M = D \otimes_{K_0} \mathbf{B}^\dagger_{\mathrm{rig},K}$, viewed as a (φ, Γ_K)-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ with trivial Γ_K-action on D.

Berger constructs another (φ, Γ_K)-module M' with $M[t^{-1}] \cong M'[t^{-1}]$, such that for n large, the t-adic filtration on $M' \otimes_{\theta_n} K(\epsilon_n)((t))$ is obtained from the t-adic filtration on $M \otimes_{\theta_n} K(\epsilon_n)((t))$ by tensoring with the provided filtration on D (viewing $K(\epsilon_n)$ as a K_0-algebra via φ^n).

Theorem (Berger)

D is weakly admissible if and only if M' is étale.

In this case, one then gets a (crystalline) Galois representation. For instance, any filtered isocrystal arising from de Rham cohomology is weakly admissible, and the resulting Galois representation computes étale cohomology (Fontaine's identification of the *mysterious functor*).

Moduli of filtrations

Assume from now on (to simplify) that $K_0 = \operatorname{Frac} W(\mathbb{F}_p^{\mathrm{alg}})$.

Fix a finite-dimensional K_0-vector space D equipped with an invertible semilinear action of φ and a multiset H of integers. Filtrations on D with weights H are parametrized by a partial flag variety $\mathcal{F}_{D,H}$ over K_0.

Rapoport and Zink proposed to pass to the rigid analytification of $\mathcal{F}_{D,H}$, cut out an open subspace containing only the weakly admissible points, and construct an étale $\mathbb{Q}_p$-local system over this subspace whose fibre over a point is the associated crystalline Galois representation (a *crystalline local system*). de Jong observed that this is more conveniently done in Berkovich's language of nonarchimedean analytic spaces.

Motivation: when all weights are 0 or 1, this construction receives a *period morphism* from a deformation space of p-divisible groups.

A word from our sponsor: Berkovich analytic spaces

Let R be a ring equipped with a nonarchimedean norm $|\bullet|$ (which we may as well take to be complete). Berkovich defines the *Gel'fand spectrum* $\mathcal{M}(R)$ as the topological space of bounded multiplicative seminorms $\alpha : R \to [0, +\infty)$, viewed as a closed (hence *compact*) subspace of $\prod_{f \in R} [0, |f|]$. This topology is generated by the open sets

$$\{\alpha \in \mathcal{M}(R) : \alpha(f) \in (a, b)\} \quad (f \in R; a, b \in \mathbb{R}).$$

The *residue field* $\mathcal{H}(\alpha)$ of α is the completion of $\mathrm{Frac}(R/\ker(\alpha))$.

As over $\mathbb{C}$, one has an analytification functor $X \mapsto X^{\mathrm{an}}$ for schemes locally of finite type over a complete nonarchimedean field (e.g., K_0). The set X^{an} includes all of the points of the rigid analytification of X (which is totally disconnected), but is much bigger (and is locally path-connected).

Admissibility and weak admissibility

Within $\mathcal{F}^{\mathrm{an}}_{D,H}$, the weakly admissible locus $\mathcal{F}^{\mathrm{wa}}_{D,H}$ is open. However, one can exhibit a (nonrigid) point within $\mathcal{F}^{\mathrm{wa}}_{D,H}$ in no neighborhood of which one can construct a suitable local system (Genestier-V. Lafforgue, Hartl).

What is needed is a more precise notion of *admissibility*, closer to slope theory of φ-modules over Robba rings. Such a notion was proposed and thoroughly analyzed by Hartl in an equal-characteristic analogue of this picture (the theory of Hodge-Pink structures).

To mirror Hartl's construction in our setting, we must use the algebra and geometry of Witt vectors to pass back and forth between positive and mixed characteristic. (Similar arguments have been given by Faltings.)

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p-typical Witt vectors

For R a perfect $\mathbb{F}_p$-algebra, let $W(R)$ denote the ring of p-typical Witt vectors. It is p-adically separated and complete with $W(R)/pW(R) \cong R$, and admits the multiplicative *Teichmüller map* $[\bullet] : R \to W(R)$. For $x = \sum_{i=0}^{\infty} p^i [\overline{x_i}]$, $y = \sum_{i=0}^{\infty} p^i [\overline{y_i}]$, we have

$$x + y = \sum_{i=0}^{\infty} p^i [P_i(\overline{x_0}^{p^{-i}}, \overline{y_0}^{p^{-i}}, \dots, \overline{x_i}^{p^{-0}}, \overline{y_i}^{p^{-0}})]$$

$$xy = \sum_{i=0}^{\infty} p^i [Q_i(\overline{x_0}^{p^{-i}}, \overline{y_0}^{p^{-i}}, \dots, \overline{x_i}^{p^{-0}}, \overline{y_i}^{p^{-0}})]$$

for certain universal polynomials P_i, Q_i over $\mathbb{Z}$. The P_i, Q_i are never written down in practice, but one sometimes needs some properties: e.g., P_i is homogeneous of degree 1 in $\overline{x_0}, \overline{y_0}, \dots, \overline{x_i}, \overline{y_i}$.

Raising and lowering operators

Theorem

For R a perfect $\mathbb{F}_p$-algebra, equip R with the trivial norm and $W(R)$ with the p-adic norm. For $\alpha \in \mathcal{M}(R)$, $\beta \in \mathcal{M}(W(R))$, define the functions

$$\lambda(\alpha)\left(\sum_i p^i[\overline{x_i}]\right) = \max_i\{p^{-i}\alpha(\overline{x_i})\}$$

$$\mu(\beta)(\overline{x}) = \beta([\overline{x}]).$$

One gets continuous maps $\lambda : \mathcal{M}(R) \to \mathcal{M}(W(R))$, $\mu : \mathcal{M}(W(R)) \to \mathcal{M}(R)$ satisfying $(\mu \circ \lambda)(\alpha) = \alpha$, $(\lambda \circ \mu)(\beta) \geq \beta$.

One even has a uniform contraction of $\mathcal{M}(W(R))$ onto the image of λ, so each $U \subseteq \mathcal{M}(R)$ has the same homotopy type as $\mu^{-1}(U)$.

Analogy: equip $R[T]$ with the Gauss norm. Then $\mathcal{M}(R[T]) \to \mathcal{M}(R)$ is a disc bundle, and the generic points of fibres form a continuous section.

Frobenius lifts and Witt vectors

Note that $\mathbf{B}_{K_0}$ is an absolutely unramified complete discretely valued field with residue field $\mathbb{F}_p^{\mathrm{alg}}((\overline{\pi}))$, and that $\varphi : \mathbf{B}_{K_0} \to \mathbf{B}_{K_0}$ lifts the absolute Frobenius on $\mathbb{F}_p^{\mathrm{alg}}((\overline{\pi}))$.

The map φ defines an embedding $\mathbf{A}_{K_0} \to W(L)$ for $L = \mathbb{F}_p^{\mathrm{alg}}((\overline{\pi}))^{\mathrm{perf}}$, under which $1 + \pi$ maps to $[1 + \overline{\pi}]$. More explicitly, one may identify $W(L)$ with the completed direct limit of

$$\mathbf{A}_{K_0} \stackrel{\varphi}{\to} \mathbf{A}_{K_0} \stackrel{\varphi}{\to} \cdots .$$

The restriction to $\mathbf{A}^{\dagger}_{K_0}$ extends to an embedding of $\mathbf{B}^{\dagger}_{\mathrm{rig},K_0}$ into a certain Fréchet completion of $W(L)[p^{-1}]$; the latter exists more generally.

Extended Robba rings

For $r > 0$, consider the Fréchet completion of $K_0[\pi^{\pm}]$ for the valuations

$$v_s\left(\sum_i c_i \pi^i\right) = \max_i\{v_p(c_i) + s\pi\} \qquad (s \in (0, r]).$$

The union over all $r > 0$ gives $\mathbf{B}^{\dagger}_{\mathrm{rig},K_0}$.

Similarly, let L be a perfect field of characteristic p carrying a valuation v (e.g., $\mathbb{F}_p^{\mathrm{alg}}((\overline{\pi}))^{\mathrm{perf}}$). For each $r > 0$, one has a valuation v_r defined by

$$v_r\left(\sum_i p^i[\overline{x_i}]\right) = \max_i\{i + rv(\overline{x_i})\}$$

on the subring $\tilde{\mathcal{R}}_L^{r,\mathrm{int}}$ of $W(R)$ on which $i + rv(\overline{x_i}) \to +\infty$ as $i \to +\infty$. Let $\tilde{\mathcal{R}}_L^r$ be the Fréchet completion of $\tilde{\mathcal{R}}_L^{r,\mathrm{int}}[p^{-1}]$ for the valuations v_s for all $s \in (0, r]$. Put $\tilde{\mathcal{R}}_L = \cup_{r>0}\tilde{\mathcal{R}}_L^r[p^{-1}]$.

More on extended Robba rings

The *extended Robba ring* $\tilde{\mathcal{R}}_L$ admits a theory of φ-modules and slopes. Again, a φ-module is étale (in the sense of having a basis on which φ acts via an invertible integral matrix) if and only if its slope is 0 and it admits no φ-submodule of negative slope.

If L is algebraically closed, one also has a Dieudonné-Manin classification, including the fact that every étale φ-module has a φ-invariant basis.

For example, if $L = \mathbb{F}_p((\overline{\pi}))^{\mathrm{alg}}$, then $\tilde{\mathcal{R}}_L$ is the "one ring" $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ used by Berger to pass from (φ, Γ)-modules to Galois representations.

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Local coordinates: an apology

When all weights are 0 or 1, Hartl proposed a notion of admissibility using a field of norms construction, and showed that the admissible locus is open.

In general, we work within a subspace of $\mathcal{F}_{D,H}^{\mathrm{an}}$ isomorphic to $\mathcal{M}(K_0[T_1^{\pm}, \dots, T_d^{\pm}])$ (for the Gauss norm). Such spaces cover $\mathcal{F}_{D,H}^{\mathrm{an}}$ because $\mathcal{F}_{D,H}$ is covered by affine spaces (big Schubert cells).

In order to glue over all of $\mathcal{F}_{D,H}^{\mathrm{an}}$, we will have to check at the end that the admissible locus, and the local system over it, do not depend on the choice of coordinates.

$(\varphi, \tilde{\Gamma})$-modules

Let S be the completion of $K_0[T_1^{\pm}, \dots, T_d^{\pm}]$ for the Gauss norm. Extend φ, Γ to $S \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ so that $\varphi(T_i) = T_i^p$, $\gamma(T_i) = T_i$. Then define the group $\tilde{\Gamma} = \Gamma \ltimes \mathbb{Z}_p^d$ acting on $S \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ so that $(e_1, \dots, e_d) \in \mathbb{Z}_p^d$ acts as the $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$-linear substitution $T_i \mapsto (1+\pi)^{e_i} T_i$.

One can define $(\varphi, \tilde{\Gamma})$-modules over $S \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, but it is more convenient to do this in geometric language. That is, view $\varphi, \tilde{\Gamma}$ as acting on the open unit disc over $\mathcal{M}(S)$, take a coherent locally free sheaf $\mathcal{E}$ on some open annulus over $\mathcal{M}(S)$ of outer radius 1, then equip $\mathcal{E}$ with isomorphisms $\varphi^*\mathcal{E} \cong \mathcal{E}, \gamma^*\mathcal{E} \cong \mathcal{E}$ on a smaller annulus.

The étale condition should still be formulated algebraically. E.g., locally on $\mathcal{M}(S)$, there should exist a basis on which φ acts via a matrix U over $S \widehat{\otimes} \mathbf{B}^{\dagger}_{K}$ such that U, U^{-1} have nonnegative p-adic valuation.

From filtrations to a $(\varphi, \tilde{\Gamma})$-module

We imitate Berger's passage from filtered isocrystals to (φ, Γ)-modules.

Start with the pullback of D to the open unit disc over $\mathcal{M}(S)$. We can modify along $\pi = 0$ so that the t-adic filtration gets tensored with the pullback of the universal filtration on $\mathcal{F}_{D,H}$. Then use φ to move this modification to $\pi = \epsilon_n^\gamma - 1$ for all $n \geq 0, \gamma \in \mathbb{Z}_p^\times$.

Put $\mathcal{E} = D \otimes_{K_0} (S \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K})$, viewed as a $(\varphi, \tilde{\Gamma})$-module with $\tilde{\Gamma}$ acting trivially on D. By the previous paragraph, there is another $(\varphi, \tilde{\Gamma})$-module $\mathcal{E}'$ with $\mathcal{E}[t^{-1}] \cong \mathcal{E}'[t^{-1}]$, such that for n large, the t-adic filtration on $\mathcal{E}' \otimes_{\theta_n} S(\epsilon_n)((t))$ is obtained from the t-adic filtration on $\mathcal{E} \otimes_{\theta_n} S(\epsilon_n)((t))$ by tensoring with the universal filtration from $\mathcal{F}_{D,H}$ (viewing $S(\epsilon_n)$ as an S-algebra via φ^n).

Following Andreatta and Brinon (and in turn Faltings), one obtains an equivalence of categories between étale local systems over $\mathcal{M}(S)$ and étale $(\varphi, \tilde{\Gamma})$-modules over $S \widehat{\otimes} \mathbf{B}^\dagger_{\mathrm{rig},K}$. However, $\mathcal{E}'$ is not étale!

Construction of the admissible locus (after Hartl)

Let $\mathfrak{o}_S$ be the subring of S of norm ≤ 1. Equip $\overline{S} = \mathbb{F}_p^{\mathrm{alg}}[\overline{T_1}^{\pm}, \dots, \overline{T_d}^{\pm}]$ and $\overline{S}' = \overline{S}[\![\overline{\pi}]\!]$ with trivial norms. We have commuting diagrams

$$\begin{array}{ccccc} \mathfrak{o}_S[\![\pi]\!] & \xrightarrow{\varphi} & \mathfrak{o}_S[\![\pi]\!] & \xrightarrow{\varphi} & \cdots \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \\ \mathfrak{o}_S(\epsilon_0) & \xrightarrow{\varphi} & \mathfrak{o}_S(\epsilon_1) & \xrightarrow{\varphi} & \cdots \end{array} \qquad \begin{array}{ccc} \mathfrak{o}_S[\![\pi]\!] & \longrightarrow & W(\overline{S}'^{,\mathrm{perf}}) \\ \downarrow{\scriptstyle \theta_0} & & \downarrow \\ \mathfrak{o}_S(\epsilon) & \longrightarrow & W(\overline{S}^{\mathrm{perf}})(\epsilon)^{\wedge} \end{array}$$

For $\alpha \in \mathcal{M}(S)$, we can extend α to $\mathfrak{o}_S(\epsilon)$ and $\mathcal{M}(W(\overline{S}^{\mathrm{perf}})(\epsilon))$. The set of such extensions $\tilde{\alpha}$ is nonempty and permuted transitively by $\tilde{\Gamma}$.

Restrict $\tilde{\alpha}$ to $\beta \in \mathcal{M}(W(\overline{S}'^{,\mathrm{perf}}))$. For $L = \mathcal{H}(\mu(\beta))^{\mathrm{perf}}$, $\mathfrak{o}_S[\![\pi]\!] \to W(\overline{S}'^{,\mathrm{perf}}) \to \tilde{\mathcal{R}}_L$ extends to $S \widehat{\otimes} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$. We say α is *admissible* if $\mathcal{E}' \otimes \tilde{\mathcal{R}}_L$ is étale.

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Overview of the construction

Theorem

(a) *The admissible locus of* $\mathcal{M}(S)$ *is an open subspace of* $\mathcal{M}(S) \cap \mathcal{F}^{\mathrm{wa}}_{D,H}$.

(b) *A rigid point is admissible if and only if it is weakly admissible.*

(c) *Over the admissible locus, there exists a crystalline local system (an étale* $\mathbb{Q}_p$*-local system whose fibre over each rigid point is the corresponding crystalline representation).*

We cannot use Andreatta-Brinon because $\mathcal{E}'$ is not étale. In fact, we will work on a certain φ-stable subspace of the open unit disc over $\mathcal{M}(S)$, which is *not* the product of an annulus with a subspace of $\mathcal{M}(S)$.

Nonproduct subspaces of $\mathcal{M}(S)$

Fix $\alpha \in \mathcal{M}(S)$ admissible. Define $\beta \in \mathcal{M}(W(\overline{S}'^{,\mathrm{perf}}))$ as before, recalling that $\overline{S}' = \overline{S}[\![\overline{\pi}]\!]$ and that φ induces $\mathfrak{o}_S[\![\pi]\!] \to W(\overline{S}'^{,\mathrm{perf}})$.

For T an open neighborhood of $\mu(\beta)$ and $r > 0$, let $\mathcal{R}^r_S(T)$ be the Fréchet completion of $\mathfrak{o}_S[\![\pi]\!][p^{-1}]$ for (the restrictions of) the seminorms $\lambda(\gamma^{s(p-1)/p}) \in \mathcal{M}(W(\overline{S}'^{,\mathrm{perf}}))$ for all $\gamma \in T$ and all $s \in (0, r]$. Put $\mathcal{R}_S(T) = \cup_{r>0}\mathcal{R}^r_S(T)$.

Since $(\lambda \circ \mu)(\bullet) \geq \bullet$, $\mathcal{R}^r_S(T)$ carries restrictions of seminorms in $\mu^{-1}(\gamma^{s(p-1)/p})$. Hence for n large, elements of $\mathcal{R}^r_S(T)$ restrict to analytic functions on an open neighborhood V of $\varphi^{-n}(\alpha)$ in $\mathcal{M}(S) \times \{\pi = \epsilon_n - 1\}$.

Cutting out the étale locus

For $s \in (0, r]$, let $v_s(\bullet)$ be the infimum of $-\log_p(\lambda(\gamma^{s(p-1)/p})(\bullet))$ over $\gamma \in T$. Let $v(\bullet)$ be the limit inferior of v_s as $s \to 0^+$.

Theorem

We can choose T so that $\mathcal{E}' \otimes \mathcal{R}_S(T)$ admits a basis on which φ acts via an invertible matrix over the subring on which v is nonnegative.

That is, we can ensure that $\mathcal{E}' \otimes \mathcal{R}_S(T)$ is *étale*.

To check this, one first approximates a good basis of $\mathcal{E}' \otimes \tilde{\mathcal{R}}_L$. One then calculates as in the (2005) proof that étaleness of a φ-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ can be checked over $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$. (In equal characteristic, this approach gives a slightly new proof of Hartl's theorem.)

Construction of the crystalline local system

Given a good choice of T, we proceed as in the nonrelative case to construct the crystalline local system near α. For each m, for some r, we can find a finite étale extension of $\mathcal{R}^r_S(T)$ over which we can trivialize the action of φ modulo p^m. By restricting along θ_n and then using $\tilde{\Gamma}$ to descend, we obtain an étale $(\mathbb{Z}_p/p^m\mathbb{Z}_p)$-local system on an open neighborhood V of α.

Because V is determined by T, it does not depend on m. We can then take the inverse limit, then pass to the associated $\mathbb{Q}_p$-local system.

Correlation at rigid points

We now check that a rigid point is admissible if and only if it is weakly admissible, and if so, identify the fibre of the local system with the crystalline representation from Berger's construction.

Suppose α is a rigid point with residue field L. Let $\tilde{\Gamma}_K$ be the stabilizer of $\tilde{\alpha}$; it is an open subgroup of $\tilde{\Gamma}$. Take the closure of $S[\pi^{\pm}]$ in $\tilde{\mathcal{R}}_L$; its $(\mathbb{Z}_p^d \cap \tilde{\Gamma}_K)$-invariants are $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ (as may be checked by extending scalars along θ_n). Likewise, the $(\mathbb{Z}_p^d \cap \tilde{\Gamma}_K)$-invariants of $\mathcal{E} \otimes \tilde{\mathcal{R}}_L$ may be identified with $M(D)$, and similarly with primes.

Independence from coordinate choices

We check that the admissible locus and the local system do not depend on the choice of coordinates. For both arguments, identify $\mathcal{M}(K_0[T_1^{\pm}, \dots, T_d^{\pm}, U_1^{\pm}, \dots, U_d^{\pm}])$ with a subspace of $\mathcal{F}^{\mathrm{an}}_{D,H} \times_{K_0} \mathcal{F}^{\mathrm{an}}_{D,H}$ and run the previous construction, this time with $\tilde{\Gamma}$ containing two copies of $\mathbb{Z}_p^d$. Embed $\mathcal{F}^{\mathrm{an}}_{D,H}$ diagonally into $\mathcal{F}^{\mathrm{an}}_{D,H} \times_{K_0} \mathcal{F}^{\mathrm{an}}_{D,H}$.

For the admissible locus, note that having a local system constructed from one set of coordinates forces admissibility in the other set (by an argument akin to the previous slide).

For the local system, restrict to the diagonal to get the isomorphism locally. The cocycle condition can be checked at rigid points, using the previous slide.