Tropical geometry of cluster varieties: a question from number theory

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Contents

Contents

A determinant identity of Desnanot–Jacobi

Take an $n \times n$ matrix with a row and column marked off along each edge:

$$\left(\begin{array}{c|ccc|c} * & * & \cdots & * & * \\ \hline * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ \hline * & * & \cdots & * & * \end{array}\right)$$

We then have

$$D \cdot C = NW \cdot SE - NE \cdot SW$$

where each term is the determinant of a certain submatrix:

- D of the full $n \times n$ matrix;
- C of the central $(n-2) \times (n-2)$-submatrix;
- NW, NE, SW, SE of the corner $(n-1) \times (n-1)$-submatrices.

More on the Desnanot–Jacobi identity

There are various ways to prove the identity

$$D \cdot C = NW \cdot SE - NE \cdot SW.$$

For instance, in a sufficiently generic situation, one may perform row reduction within the central submatrix without changing any of the terms; one thus reduces to the easy case

$$\left(\begin{array}{c|c|c} * & 0 & * \\ \hline 0 & I_{n-2} & 0 \\ \hline * & 0 & * \end{array}\right).$$

A more modern approach would be to interpret the identity as a Plücker relation on a Grassmannian. Hold that thought...

Dodgson's condensation method

Let M be an $n \times n$ matrix over an integral domain R. Charles Dodgson[1] proposed a method for computing $\det(M)$ based on the Desnanot–Jacobi identity: compute a sequence of matrices $M^{(1)}, \dots, M^{(n)}$ (which I'll call *layers*) where $M^{(k)}$ consists of the connected $k \times k$ minors. The identity implies that each layer can be computed from the preceding two.

This (if it works) is an $O(n^3)$ algorithm like Gaussian elimination, but has some intriguing benefits: it is easily parallelizable with low communication, and all intermediate terms lie in R rather than $\mathrm{Frac}(R)$.

However, it doesn't always work: in the identity

$$D \cdot C = NW \cdot SE - NE \cdot SW$$

we might at some point have $C = 0$, in which case we cannot solve for D.

[1] a/k/a Lewis Carroll, of *Alice in Wonderland* fame.

Contents

The work of Robbins

Motivated by **REDACTED**, Mills, Robbins, and Rumsey began to investigate the condensation method in the early 1980s. This led to some remarkable connections with enumerative combinatorics, including a proof of Macdonald's conjecture on plane partitions, and the related enumeration of alternating sign matrices.

Robbins was particularly interested in condensation over finite fields. This has an obvious difficulty: zero minors are far more likely to occur than over an infinite domain. One could try lifting, say from $\mathbb{F}_p$ to $\mathbb{Z}$, but this creates an undesirable coefficient explosion.

Instead, Robbins proposed lifting to a complete DVR, say from $\mathbb{F}_p$ to $\mathbb{Z}_p$. As with $\mathbb{R}$, a complete DVR is an *inexact ring* from the point of view of machine computing; one must choose a scheme for computing systematically with finite approximations (and some roundoff errors).

p-adic floating-point arithmetic

Since condensation involves division, we cannot work in a fixed quotient of our complete DVR. Instead, we use the natural analogue of *floating-point arithmetic*. For simplicity, let me describe only the case of $\mathbb{Z}_p$.

Each p-adic floating-point number consists of a power of p (the *exponent*) times a p-adic unit reduced modulo a fixed power of p (the *mantissa*). This represents a certain p-adic ball; we perform arithmetic as if this entire ball were identified with some particular representative.

There is *no loss of precision* when multiplying two floating-point numbers: whatever "true" values we have in mind, we get the correct ball around the product. However, addition creates a loss of precision when the valuation of $x + y$ is greater than the (common) valuation of x and y; one lacks enough information to renormalize the mantissa.

At the time of Robbins, this was still largely theoretical. Nowadays, it is implemented in `Pari`, `Magma`, `Sage`...

A numerical observation

If one works consistently with mantissas of m digits, the final result of a computation will generally only be accurate to some smaller number of digits of (relative) precision. We refer to the difference as *precision loss*.

Although nonarchimedean roundoff errors do not compound like as their archimedean counterparts, one still generally observes steady degradation of precision over the course of a computation, as the effect of *individual* roundoff errors is progressively magnified.

Also, without an external precision analysis, one cannot easily detect the precision loss from the returned output. That is, one cannot directly judge the quality of the reported answer!

For condensation, Robbins conjectured a different behavior: the precision loss can be bounded precisely, by the maximum valuation of any of the computed minors. This is supported by *vast* numerical evidence.

A mathematical reformulation

Let's reformulate this conjecture in more concrete terms, and generalize to a general DVR called R (with valuation v).

Let M be an $n \times n$ matrix over R. Let $M_{ij}^{(k)}$ denote the minor of M consisting of rows $i, \ldots, i+k-1$ and columns $j, \ldots, j+k-1$. The matrices $M^{(0)}, \ldots, M^{(n)}$ satisfy the condensation recurrence

$$M_{ij}^{(k+1)} M_{(i+1)(j+1)}^{(k-1)} = M_{ij}^{(k)} M_{(i+1)(j+1)}^{(k)} - M_{i(j+1)}^{(k)} M_{(i+1)j}^{(k)}.$$

Let $\tilde{M}^{(0)}, \ldots, \tilde{M}^{(n)}$ be a sequence, with the same first two terms, with

$$\tilde{M}_{ij}^{(k+1)} \tilde{M}_{(i+1)(j+1)}^{(k-1)} = *\tilde{M}_{ij}^{(k)} \tilde{M}_{(i+1)(j+1)}^{(k)} - *\tilde{M}_{i(j+1)}^{(k)} \tilde{M}_{(i+1)j}^{(k)}$$

where each $*$ is a (different) element of R with $v(1-*) \geq m$.

The conjecture of Robbins, and a partial result

Conjecture (after Robbins)

We have $v(M^{(n)} - \tilde{M}^{(n)}) \geq m - b$ *where* $b = \max_{i,j,k}\{v(M_{ij}^{(k)})\}$.

Theorem (B-K)

We have $v(M^{(n)} - \tilde{M}^{(n)}) \geq m - 3b$ *where* $b = \max_{i,j,k}\{v(M_{ij}^{(k)})\}$.

In fact, this conjecture generalizes to a corresponding statement about recurrences derived from cluster algebras, and the theorem is based on the proof of the Laurent phenomenon for such algebras. Our methods to date have been predominantly algebraic; it is an open question whether geometry of tropicalizations can be exploited.

Contents

Some combinatorial data

Fix a positive integer n. Let Γ_n be the Cayley graph of the free product of n copies of $\mathbb{Z}/2\mathbb{Z}$; it is a n-regular tree with edges labeled $\{1, \dots, n\}$.

We consider a family of $n \times n$ matrices $\{B_v\}_{v \in \Gamma_n}$ called *seeds*, which satisfy a relation called *mutation*: for any vertices v, w which are endpoints of an edge labeled k,

$$B_{w,ij} = \begin{cases} -B_{v,ij} & \text{if } i = k \text{ or } j = k \\ B_{v,ij} + \frac{1}{2}(|B_{v,ik}|B_{v,kj} + B_{v,ik}|B_{v,kj}|) & \text{otherwise;} \end{cases}$$

this family is determined by any one B_v. To make this relation symmetric, each B_v must be *sign-skew-symmetric* (i.e., the matrix $(\mathrm{sgn}(B_{v,ij}))_{i,j}$ is skew-symmetric); this holds if one B_v is *skew-symmetrizable* (i.e., conjugate to a skew-symmetric matrix via a positive diagonal matrix).

From seeds to cluster algebras

Let K be a field (for convenience; one can also handle more general base rings and semirings). For each $v \in \Gamma_n$, form the field $K(x_{v,1}, \dots, x_{v,n})$. For v, w which are endpoints of an edge labeled k, identify $K(x_{v,1}, \dots, x_{v,n})$ with $K(x_{w,1}, \dots, x_{w,n})$ so that the *exchange relation* holds:

$$x_{v,i} = x_{w,i} \quad (i \neq k), \qquad x_{v,k}x_{w,k} = \prod_{j \neq k : B_{v,kj} > 0} x_{v,j}^{B_{v,kj}} + \prod_{j \neq k : B_{v,kj} < 0} x_{v,j}^{-B_{v,kj}}.$$

The *cluster algebra* defined by the chosen seeds is the union of the rings $K[x_{v,1}, \dots, x_{v,n}]$ over all $v \in \Gamma_n$.

Theorem (Fomin–Zelevinsky caterpillar lemma)

The cluster algebra is contained in $K[x_{v,1}^{\pm}, \dots, x_{v,n}^{\pm}]$ for each $v \in \Gamma_n$. (In general, it is strictly smaller than the intersection of these rings.)

Floating-point errors

Let R be a DVR with valuation v. Let $w_0, \dots, w_\ell$ be a path in Γ_n. Specialize $x_{w_0,1}, \dots, x_{w_0,n}$ to units in R; by the caterpillar lemma,

$$x_{w_i,j} \in R \qquad (i = 1, \dots, \ell; j = 1, \dots, n).$$

Let k_i be the label of the edge from w_{i-1} to w_i; then

$$x_{w_{i-1},k_i} x_{w_i,k_i} = \prod_{j \neq k_i : B_{w_{i-1},k_i j} > 0} x_{w_{i-1},j}^{B_{w_{i-1},k_i j}} + \prod_{j \neq k_i : B_{w_{i-1},k_i j} < 0} x_{w_{i-1},j}^{-B_{w_{i-1},k_i j}}.$$

Define modified values $\tilde{x}_{w_i,j}$ so that $\tilde{x}_{w_0,j} = x_{w_0,j}$ and

$$\tilde{x}_{w_{i-1},k_i} \tilde{x}_{w_i,k_i} = * \prod_{j \neq k_i : B_{w_{i-1},k_i j} > 0} \tilde{x}_{w_{i-1},j}^{B_{w_{i-1},k_i j}} + * \prod_{j \neq k_i : B_{w_{i-1},k_i j} < 0} \tilde{x}_{w_{i-1},j}^{-B_{w_{i-1},k_i j}}.$$

where each $*$ is a (different) element of R with $v(1 - *) \geq m$.

The Robbins phenomenon

Conjecture

For all j, $v(x_{w_\ell,j} - \tilde{x}_{w_\ell,j}) \geq m - b$ where $b = \max_{i,j}\{v(x_{w_i,j})\}$.

This has been tested in many cases, including with random seeds.

Theorem (B-K)

For all j, $v(x_{w_\ell,j} - \tilde{x}_{w_\ell,j}) \geq m - cb$ where $b = \max_{i,j}\{v(x_{w_i,j})\}$ and c is a positive integer depending only on the seeds and the path.

More precisely, we may replace cb with the maximum of

$$v(x_{w_{i-1},k_i}) + \sum_{j \neq k_i : \mathrm{sgn}(B_{w_{i-1},k_i j} = \epsilon)} v(x_{w_{i-1},j})$$

over $i \in \{1, \ldots, n\}$ and $\epsilon \in \{\pm 1\}$. (That is, take the moving variable plus all of the variables appearing in one of the two monomials.)

Contents

Condensation as a cluster algebra

The importance of cluster algebras is that they incorporate *many* preexisting examples from disparate contexts, thus inducing people who previously had no shared interests to communicate!

For example, the homogeneous coordinate ring of a Grassmannian can be interpreted as a cluster algebra, with the exchange relations reproducing Plücker relations. This example gives rise to condensation: one has a path where, at certain steps along the way, the cluster variables correspond to all of the minors in two consecutive labels.

Besides this example, our typical examples are not the most common from other points of view (e.g., triangulations of surfaces). We prefer examples which correspond to simple algebraic recurrences, as these are easy to test. (That said, we have done some numerical experiments involving random skew-symmetrizable seeds and random walks on Γ_n.)

Example: the Somos-4 recurrence

Let $x_0, \ldots, x_3$ be units in R. Define $x_4, \ldots, x_\ell$ by the recurrence

$$x_i x_{i-4} = x_{i-1} x_{i-3} + x_{i-2}^2.$$

As usual, let $\tilde{x}_i$ be a modified recurrence with the same initial terms. Then

$$v(x_\ell - \tilde{x}_\ell) \geq m - \max_i \{v(x_{i-4}) + \max\{v(x_{i-1}) + v(x_{i-3}), v(x_{i-2})\}\}.$$

However, it is easy to show by induction on i that at most one of $x_{i-4}, x_{i-3}, x_{i-2}, x_{i-1}$ is *not* a unit in R. So we actually deduce the full Robbins conjecture in this case. (See Caruso–Roe–Vaccon for another proof using *differential precision analysis*.)

Not an example: the Somos-6 recurrence

Let $x_0, \dots, x_5$ be units in R. Define $x_6, \dots, x_\ell$ by the recurrence

$$x_i x_{i-6} = x_{i-1} x_{i-5} + x_{i-2} x_{i-4} + x_{i-3}^2.$$

This is not a cluster example (since the recurrence is trinomial rather than binomial), but still exhibits the *Laurent phenomenon*; in particular, $x_\ell \in R$.

As usual, let $\tilde{x}_i$ be a modified recurrence with the same initial terms. Then

$$v(x_\ell - \tilde{x}_\ell) \geq m - c \max_i \{v(x_i)\}$$

where we can prove $c = 5$, and examples suggest that $c = 2$ is best possible. We have counterexamples against the inequality for $c = 1$.

Lam–Pylyavskyy have defined *LP algebras* by analogy with cluster algebras. Our theorem adapts to these, but it is unclear how to formulate a conjecture that optimizes c.

Example: Somos-6 with no middle term

Let $x_0, \dots, x_5$ be units in R. Define $x_6, \dots, x_\ell$ by the recurrence

$$x_i x_{i-6} = x_{i-1} x_{i-5} + x_{i-2} x_{i-4}.$$

As usual, let $\tilde{x}_i$ be a modified recurrence with the same initial terms. Then

$$v(x_\ell - \tilde{x}_\ell) \geq m - \max_i \{v(x_{i-6}) + \max\{v(x_{i-1}) + v(x_{i-5}), v(x_{i-2}) + v(x_{i-4})\}\}.$$

This is again a cluster recurrence, so in the bound

$$v(x_\ell - \tilde{x}_\ell) \geq m - c \max_i \{v(x_i)\}$$

we expect to achieve $c = 1$; however, the argument for Somos-4 does not carry over. But maybe some more sophisticated argument about the valuations in a slightly longer subsequence would help?

Contents

Introducing extra variables

One can formulate a stronger conjecture, and establish some partial results, which does not depend on a discrete valuation. In this setup, each error term $*$ is replaced by $1+\epsilon$ where ϵ is a (suitably labeled) polynomial variable. For instance, in the cluster recurrence

$$\tilde{x}_{w_{i-1},k_i}\tilde{x}_{w_i,k_i} = * \prod_{j\neq k_i : B_{w_{i-1},k_i j}>0} \tilde{x}_{w_{i-1},j}^{B_{w_{i-1},k_i j}} + * \prod_{j\neq k_i : B_{w_{i-1},k_i j}<0} \tilde{x}_{w_{i-1},j}^{-B_{w_{i-1},k_i j}}.$$

we may replace the two $*$ terms by $1+\epsilon_{i,+}$ and $1+\epsilon_{i,-}$.

The Laurent phenomenon will now fail: it is not the case that

$$\tilde{x}_{w_\ell,j} \in R[\tilde{x}_{w_0,i}^{\pm}][\epsilon_{i,\pm}].$$

However, we can prove that this holds if we adjoin not $\epsilon_{i,+},\epsilon_{i,-}$ but

$$\frac{\epsilon_{i,+}}{\tilde{x}_{w_{i-1},k}\prod_{j\neq k_i : B_{w_i,k_i j}<0} x_{w_{i-1},j}}, \frac{\epsilon_{i,-}}{\tilde{x}_{w_{i-1},k}\prod_{j\neq k_i : B_{w_i,k_i j}>0} x_{w_{i-1},j}}$$

(modulo a suitable localization, or use power series variables instead).

An algebraic Robbins conjecture

In the previous notation, we conjecture more: instead of adjoining $\epsilon_{i,+}$ divided by a product of variables, it suffices to adjoin $\epsilon_{i,+}$ divided by each variable individually. This would then explain the constant $c = 1$ in the Robbins conjecture.

We have limited direct evidence for this stronger conjecture. For one, our experiments cannot distinguish between this statement and the weaker one where we replace the containing ring with its integral closure.

One case we do know: for Somos-4, among four consecutive terms, any two generate the unit ideal in R. From this, it follows easily that the conjecture is equivalent to the previous theorem. (The method of Caruso–Roe–Vaccon does not promote to this context.)

A first-order result

Our last evidence for the Robbins phenomenon is the following statement.

Theorem (B-P, in progress)

The algebraic Robbins conjecture for cluster algebras (previous slide) holds **to first order**, *i.e., modulo the ideal generated by any product of two ϵ factors with arbitrary denominator.*

This in particular implies that the original Robbins conjecture holds when the working precision m is sufficiently large relative to the valuations of the terms in the sequence.

It may be possible to use similar methods to establish higher-order statements, and thus obtain the algebraic Robbins conjecture in the power series formulation.