Drinfeld's lemma for F-isocrystals

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*The UCSD campus sits on the ancestral homelands of the Kumeya'ay Nation; the Kumeya'ay people continue to have an important and thriving presence in the region.

Shtukas and geometric Langlands

Let X be a curve over $\mathbb{F}_p$. To prove the Langlands correspondence for $\mathrm{GL}(2)$ over $k(X)$, Drinfeld considered moduli spaces of shtukas as geometric analogues of modular curves/Shimura varieties.

For S an $\mathbb{F}_p$-scheme, a S-**shtuka** on X is (roughly) a vector bundle on $X \times_{\mathbb{F}_p} S$ equipped with a rational map from this bundle to its φ_S-pullback (where φ_S denotes absolute Frobenius on S, fixing X).

The moduli space of shtukas admits **Hecke correspondences** corresponding to points of X, coming from **modifying** a shtuka along a point of X (by rescaling the rational map).

A similar construction was used by L. Lafforgue to extend Drinfeld's work to $\mathrm{GL}(n)$. This made heavy use of **automorphic trace formulas**.

Geometric Langlands via excursion operators

Recently, V. Lafforgue gave a more geometric, less trace-theoretic version of Drinfeld's method that can handle general reductive groups, building on the **geometric Satake equivalence** of Mirković–Vilonen.

At a key stage (the construction of **excursion operators**), this depends on an old idea of Drinfeld: the relationship between X and the formal quotient $(X \times_{\mathbb{F}_p} k)/\varphi_k$, where k is an algebraically closed field.

This relationship ("Drinfeld's lemma") takes a variety of forms. In its original form, it expresses a comparison of (profinite) étale fundamental groups and of lisse/constructible ℓ-adic sheaves for any prime $\ell \neq p$.

Variants of Drinfeld's lemma

In this talk, we focus on the situation where $\ell = p$. That is, we trade étale cohomology for Berthelot's **rigid cohomology**, in which the analogue of lisse sheaves are **overconvergent F-isocrystals**. The analogue of constructible sheaves are **arithmetic $\mathcal{D}$-modules**, but these will mostly lurk in the background.

Aside: there is another form of Drinfeld's lemma involving **perfectoid spaces**. Ongoing work of Fargues–Scholze applies this to the local Langlands correspondence in mixed characteristic $(0, p)$, again with ℓ-adic coefficients for $\ell \neq p$. Work of Carter–KSK–Zábrádi on **multivariate (φ, Γ)-modules** suggests a link with the $\ell = p$ case à la Colmez's p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Setup: a formal quotient by Frobenius

Throughout this section...

$X =$ a scheme over $\mathbb{F}_p$
$k =$ an algebraically closed field of characteristic p
$X_k = X \times_{\mathbb{F}_p} k$
$\varphi_k =$ the pullback to X_k of the absolute Frobenius on $\operatorname{Spec} k$

We will consider "X_k/φ_k" as a formal quotient: an object of some type over X_k/φ_k is an object of the same type over X_k equipped with an isomorphism with its φ_k-pullback.

Coherent sheaves: the original Drinfeld's Lemma

Theorem (Drinfeld, Lau)

For $X/\mathbb{F}_p$ (finite type and) projective, the base extension functor

$$\text{(coherent sheaves on } X) \to \text{(coherent sheaves on } X_k/\varphi_k)$$

is an equivalence of categories and preserves cohomology. (In the latter case, this means the hypercohomology of the complex $\mathcal{E} \stackrel{\varphi_k - 1}{\to} \mathcal{E}$.)

When $X = \operatorname{Spec} \mathbb{F}_p$, this says that a finite-dimensional k-vector space with a semilinear (bijective) φ_k-action has a fixed basis. This is nonabelian Artin–Schreier, a/k/a Katz–Lang (see SGA 7, XXII).

The general case follows using the fact that a coherent sheaf $\mathcal{E}$ on X (resp. X_k) can be recovered from its spaces of sections $H^0(X, \mathcal{E}(n))$ (resp. $H^0(X_k, \mathcal{E}(n))$) for $n \gg 0$.

Finite étale covers

Let $\mathbf{FEt}(X)$ be the category of finite étale schemes over X.

Corollary

The base extension functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X_k/\varphi_k)$ *is an equivalence.*

This formally reduces to the case where X is affine and of finite type over $\mathbb{F}_p$. In this case, one can choose a projective compactification Y of X; any object of $\mathbf{FEt}(X_k/\varphi_k)$ extends to a finite normal cover of Y_k/φ_k, which by the theorem comes from some finite cover of Y.

Profinite fundamental groups and lisse sheaves

Corollary

For X connected, X_k/φ_k is connected and for any geometric point $\overline{x} \to X_k$, $\pi_1^{\mathrm{prof}}(X_k/\varphi_k, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X, \overline{x})$.

Warning: in general $\pi_0(X_k) \neq \pi_0(X)$. For example, if $X = \operatorname{Spec} \ell$ is a geometric point, $\pi_0(X_k) \cong \widehat{\mathbb{Z}}$ indexed by identifications of the copies of $\overline{\mathbb{F}}_p$ in k and ℓ; but φ_k acts on $\pi_0(X_k)$ by translation by $\mathbb{Z}$.

Corollary

For $\ell \neq p$ prime, the pullback functor

$$\{\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X\} \to \{\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X_k/\varphi_k\}$$

is an equivalence of categories and preserves cohomology.

Open subschemes and étale sheaves

Corollary

The quasicompact open subschemes of X_k/φ_k are exactly the pullbacks of the quasicompact open subschemes of X.

Quasicompactness lets us reduce to the case where X is affine and of finite type over $\mathbb{F}_p$. In this case, choose a projective compactification Y; given an open subscheme U of X_k invariant under φ_k, apply the theorem to the ideal sheaf on Y_k defining the reduced complement of U.

Corollary

For X of finite type over $\mathbb{F}_p$ and $\ell \neq p$ prime, the pullback functor

$$\{\text{constructible } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X\} \to \{\text{constructible } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X_k/\varphi_k\}$$

is an equivalence of categories and preserves cohomology.

Products of two (or more) fundamental groups

Corollary

For X_1, X_2 two connected $\mathbb{F}_p$-schemes, one of which is qcqs, put $X = X_1 \times_{\mathbb{F}_p} X_2$ and let $\varphi_1, \varphi_2 : X \to X$ be the partial Frobenius maps. Then X/φ_2 is connected, and for any geometric point $\overline{x} \to X$,

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x}).$$

This follows from the case where X_2 is a geometric point using the homotopy exact sequence for a fibration (SGA 1, X).

A similar conclusion holds for n schemes, all but one of which are qcqs.

Convergent F-isocrystals

Although we eventually want to consider **overconvergent F-isocrystals**, we need to start with a slightly simpler definition.

Let X be a smooth affine scheme over a perfect field k of characteristic p. Fix a formal scheme P smooth over $W(k)$ with $P_k \cong X$ and a lift σ of φ_X to P.

A **convergent F-isocrystal** on X is a finite projective module over $\Gamma(P, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k)[p^{-1}]$-linear connection and a horizontal isomorphism with its σ-pullback.

The resulting $\mathbb{Q}_p$-linear tensor category $\mathbf{F\text{-}Isoc}(X)$ does not depend[†] on P or σ, and extends by glueing to general smooth X. The σ-action then coincides with the functorial φ_X-action.

[†] This follows by comparison with Ogus's site-theoretic definition.

Newton polygons

For $\mathcal{E} \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$ and $\overline{x} \to X$ a geometric point lying over $x \in X$, we may pull back $\mathcal{E}$ to $\mathbf{F}\text{-}\mathbf{Isoc}(\overline{x})$ and apply the Dieudonné–Manin classification: that pullback decomposes as $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ where for $d = \frac{r}{s} \in \mathbb{Q}$ in lowest terms, $\mathcal{E}_d$ admits a basis killed by $\varphi_X^s - p^r$.

Make the (convex) Newton polygon having slope d with multiplicity $\operatorname{rank} \mathcal{E}_d$ for all d; this depends only on x, not on $\overline{x}$, and is denoted $\mathrm{NP}(\mathcal{E}, x)$.

Theorem (Grothendieck–Katz)

The Newton polygon function $\mathrm{NP}(\mathcal{E}, \bullet)$ on $|X|$ is upper semicontinuous.

For example, the middle cohomology of the universal elliptic curve over a modular curve has generic Newton slopes $0, 1$ (ordinary), but at isolated points it jumps to $\frac{1}{2}, \frac{1}{2}$ (supersingular).

Slope filtrations

Theorem (Katz)

If $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant on $|X|$, then $\mathcal{E}$ admits a filtration

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

in which $\mathcal{E}_i/\mathcal{E}_{i-1}$ has all Newton slopes equal to μ_i, and $\mu_1 < \cdots < \mu_l$.

We say that $\mathcal{E}$ is **unit-root** (or **étale**) if $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant on $|X|$ with all slopes equal to 0.

Theorem (Katz, Crew)

The functor $\mathcal{E} \mapsto \mathcal{E}^{\varphi_X}$ defines an equivalence between the (full) category of unit-root objects of $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ and the category of lisse $\mathbb{Q}_p$-sheaves on X.

The structure of convergent F-isocrystals

Given $\mathcal{E} \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$, there exists an open dense subscheme U of $\mathcal{E}$ on which $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant. We then obtain a slope filtration in $\mathbf{F}\text{-}\mathbf{Isoc}(U)$. Each successive quotient is, up to a twist[‡], associated to some lisse $\mathbb{Q}_p$-sheaf. However, the extensions between these pieces do not come from lisse sheaves, and so require some analysis directly in $\mathbf{F}\text{-}\mathbf{Isoc}(U)$.

For example, suppose that there are two distinct slopes, so that we have an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0.$$

The extension class belongs to H^1 of the object $\mathcal{E}_2^\vee \otimes \mathcal{E}_1$, which is not unit-root.

[‡]This twist might be fractional, in which case we must either replace φ_X with a power or extend coefficients from $\mathbb{Q}_p$ to a ramified extension.

Convergent Φ-isocrystals

For $i = 1, 2$, let X_i be a smooth affine scheme over a perfect field k_i of characteristic p. Fix a formal scheme P_i smooth over $W(k_i)$ with $(P_i)_{k_i} \cong X_i$ and a lift σ_i of φ_{X_i} to P_i.

A **convergent Φ-isocrystal** on $X = X_1 \times_{\mathbb{F}_p} X_2$ is a finite projective module over $\Gamma(P_1 \times_{\mathbb{Z}_p} P_2, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k_1 \otimes_{\mathbb{F}_p} k_2)[p^{-1}]$-linear connection and commuting horizontal isomorphisms with its σ_i-pullbacks. Let $\Phi\,\mathbf{Isoc}(X)$ be the resulting category; again, this is functorially independent of any choices.

We may make a similar definition for an n-fold product for $n > 2$.

Pullback

There is a natural pullback functor from $\mathbf{F}\text{-}\mathbf{Isoc}(X_1)$ to $\Phi\,\mathbf{Isoc}(X)$. When X_2 is a geometric point, this admits the one-sided inverse

$$\mathcal{E} \in \Phi\,\mathbf{Isoc}(X) \mapsto \mathcal{E}^{\varphi_2} \in \mathbf{F}\text{-}\mathbf{Isoc}(X_1)$$

but is **not** an equivalence (see below).

However, for $\mathcal{E}$ as above, the sequence

$$0 \to \mathcal{E}^{\varphi_2} \to \mathcal{E} \stackrel{\varphi_2 - 1}{\to} \mathcal{E} \to 0$$

is exact. This implies...

Lemma

For X_2 a geometric point, pullback from $\mathbf{F}\text{-}\mathbf{Isoc}(X_1)$ to $\Phi\,\mathbf{Isoc}(X)$ preserves cohomology.

Total Newton polygons

If $k_1 = \mathbb{F}_p$, then X is itself smooth over k_2, and an object of $\Phi\,\mathbf{Isoc}(X)$ is just an object of $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ equipped with an isomorphism with its φ_2-pullback.

In fact, one can define $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ so that this remains true for arbitrary k_1, k_2. We then have a functor from $\Phi\,\mathbf{Isoc}(X)$ to $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ that keeps the action of $\varphi = \varphi_1 \circ \varphi_2$.

Theorem

*Suppose that X_2 is a geometric point. For $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$, the **total Newton polygon** of $\mathcal{E}$ (i.e., the Newton polygon of the image object in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$), as a function on $|X|$, factors through $|X_1|$.*

Idea of proof: apply Drinfeld's lemma to the stratification from Grothendieck–Katz.

Relative Dieudonné–Manin

Theorem

Suppose that X_2 *is a geometric point. Then any* $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$ *decomposes as* $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ *where for* $d = \frac{r}{s} \in \mathbb{Q}$ *in lowest terms,* $\mathcal{E}_d^{\varphi_2^s - p^r} \in \mathbf{F}\text{-}\mathbf{Isoc}(X_1)$. *In particular,* $\mathcal{E}$ *is a pullback from* $\mathbf{F}\text{-}\mathbf{Isoc}(X_1)$ *iff* $\mathcal{E} = \mathcal{E}_0$.

Idea of proof: first do the case where the total Newton polygon is constant, by treating the steps of the slope filtration using lisse sheaves and then applying preservation of cohomology. Then use:

Theorem (after de Jong, KSK)

For $U_i \subseteq X_i$ *open dense and* $U = U_1 \times_{\mathbb{F}_p} U_2$, $\Phi\,\mathbf{Isoc}(X) \to \Phi\,\mathbf{Isoc}(U)$ *is fully faithful.*

Products of two (or more) schemes

Theorem (not just a corollary!)

Any irreducible $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$ is a subobject of the form $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ for some $\mathcal{E}_i \in \mathbf{F}\text{-}\mathbf{Isoc}(X_i)$.

Again, we first do the case where the total Newton polygon is constant, then use the full faithfulness of restriction.

When $k_1 = k_2 = \mathbb{F}_p$, this can be reformulated as an analogue of the isomorphism

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x})$$

in terms of Tannakian fundamental groups (Daxin Xu).

A similar statement holds for n-fold products for $n > 2$.

Overconvergent F-isocrystals

Convergent F-isocrystals are not the "correct" p-adic analogue of lisse sheaves because their cohomology (of the underlying isocrystal without Frobenius) is not finite-dimensional.

This can be fixed by considering **overconvergent F-isocrystals**. For X smooth over a perfect field k, the category $\mathbf{F\text{-}Isoc}^\dagger(X)$ can be described by modifying the construction of $\mathbf{F\text{-}Isoc}(X)$ replacing the formal scheme P with a **weak formal scheme**. For instance, if $X = \mathbb{A}^n_k$, then instead of the formal scheme $P = \widehat{\mathbb{A}}^n_{W(k)}$ with

$$\Gamma(P, \mathcal{O}) = W(k)\langle T_1, \dots, T_n\rangle$$

(power series convergent on the closed unit disc), we take the weak formal scheme $P^\dagger$ with

$$\Gamma(P^\dagger, \mathcal{O}) = W(k)\langle T_1, \dots, T_n\rangle^\dagger = \varinjlim_{r>1} W(k)\langle T_1/r, \dots, T_n/r\rangle^\dagger$$

(power series convergent on some closed polydisc with radii > 1).

Overconvergent Φ-isocrystals

For $X = X_1 \times_{\mathbb{F}_p} X_2$ with X_i smooth over a perfect field k_i, we may similarly modify the definition of $\Phi\,\mathbf{Isoc}(X)$ to obtain a category $\Phi\,\mathbf{Isoc}^\dagger(X)$ of **overconvergent Φ-isocrystals**.

Theorem (after de Jong, KSK)

The functor $\Phi\,\mathbf{Isoc}^\dagger(X) \to \Phi\,\mathbf{Isoc}(X)$ is fully faithful.

Warning: this does not imply that subobjects lift from $\Phi\,\mathbf{Isoc}(X)$ to $\Phi\,\mathbf{Isoc}^\dagger(X)$. In particular, even if an object of $\Phi\,\mathbf{Isoc}^\dagger(X)$ has constant total Newton polygon, it does not typically admit a slope filtration.

We may make a similar definition for an n-fold product for $n > 2$.

Convergent to overconvergent

Using the full faithfulness of restriction, we obtain the following results by easy reductions to their convergent analogues.

Theorem

Suppose that X_2 is a geometric point. Then any $\mathcal{E} \in \Phi\,\mathbf{Isoc}^\dagger(X)$ decomposes as $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ where for $d = \frac{r}{s} \in \mathbb{Q}$ in lowest terms, $\mathcal{E}_d^{\varphi_2^s - p^r} \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_1)$. In particular, $\mathcal{E}$ is a pullback from $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ iff $\mathcal{E} = \mathcal{E}_0$.

Theorem

Any irreducible $\mathcal{E} \in \Phi\,\mathbf{Isoc}^\dagger(X)$ is a subobject of the form $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ for some $\mathcal{E}_i \in \mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X_i)$.

Beyond the smooth case

The previous results included some smoothness restrictions. One can formally promote these theorems with “smooth” replaced by “locally of finite type” using de Jong’s theorem to construct simplicial resolutions.

The same logic allows promotion from schemes to stacks. This is important for the application to the Langlands correspondence because moduli spaces of shtukas are generally not schemes.

Arithmetic $\mathcal{D}$-modules

A p-adic analogue of constructible ℓ-sheaves is used by T. Abe to adapt L. Lafforgue's proof of the Langlands correspondence for $\mathrm{GL}(n)$ to p-adic coefficients.

Given such a "sheaf" on $X_k = X \times_{\mathbb{F}_p} k$ where X is a scheme (or an algebraic stack) of finite type over $\mathbb{F}_p$, there is a maximal open dense subspace U_k on which this object restricts to an overconvergent F-isocrystal. If we start with an object on X_k/φ_k, then (by Drinfeld's lemma) U_k is the base extension of an open subspace U of X.

Is this enough to adapt V. Lafforgue's construction to p-adic coefficients? Time will tell...

Isocrystals without Frobenius structure

One can also consider convergent and overconvergent isocrystals without Frobenius structure (although the definition is a bit different).

For $X = X_1 \times_{\mathbb{F}_p} X_2$, are there similar results relating isocrystals on X_1 and X_2 to isocrystals on X equipped with a partial Frobenius action? Our techniques cannot touch this question (even when X_2 is a geometric point).

Further reading (with links)

Anne T. Carter, KSK, and Gergely Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules, arXiv.

KSK, Sheaves, stacks, and shtukas, Arizona Winter School 2017 lectures.

KSK, Notes on isocrystals, arXiv.

KSK, Étale and crystalline companions, I (arxiv), II (arXiv).

KSK, Several forms of Drinfeld's lemma, RAMpAGe seminar talk.

Eike Lau, On generalised $\mathcal{D}$-shtukas, PhD thesis (Bonn, 2004).

Peter Scholze and Jared Weinstein, *Berkeley Lectures on p-adic Geometry*.