Drinfeld's lemma for F-isocrystals

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These slides can be found at http://kskedlaya.org/slides/.

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*The UCSD campus sits on the ancestral homelands of the Kumeya'ay Nation; the Kumeya'ay people continue to have an important and thriving presence in the region.

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Shtukas and geometric Langlands

Let X be a curve over $\mathbb{F}_p$. To prove the Langlands correspondence for $\mathrm{GL}(2)$ over $k(X)$, Drinfeld considered moduli spaces of shtukas as geometric analogues of modular curves/Shimura varieties.

For S an $\mathbb{F}_p$-scheme, a S-**shtuka** on X is (roughly) a vector bundle on $X \times_{\mathbb{F}_p} S$ equipped with a rational map from this bundle to its φ_S-pullback (where φ_S denotes absolute Frobenius on S, fixing X).

The moduli space of shtukas admits **Hecke correspondences** corresponding to points of X, coming from **modifying** a shtuka along a point of X (by rescaling the rational map).

A similar construction was used by L. Lafforgue to extend Drinfeld's work to $\mathrm{GL}(n)$. This made heavy use of **automorphic trace formulas**.

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Geometric Langlands via excursion operators

Recently, V. Lafforgue gave a more geometric, less trace-theoretic version of Drinfeld's method that can handle general reductive groups, building on the **geometric Satake equivalence** of Mirković–Vilonen.

At a key stage (the construction of **excursion operators**), this depends on an old idea of Drinfeld: the relationship between X and the formal quotient $(X \times_{\mathbb{F}_p} k)/\varphi_k$, where k is an algebraically closed field.

This relationship ("Drinfeld's lemma") takes a variety of forms. In its original form, it expresses a comparison of (profinite) étale fundamental groups and of lisse/constructible ℓ-adic sheaves for any prime $\ell \neq p$.

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Variants of Drinfeld's lemma

In this talk, we focus on the situation where $\ell = p$. That is, we trade étale cohomology for Berthelot's **rigid cohomology**, in which the analogue of lisse sheaves are **overconvergent F-isocrystals**. The analogue of constructible sheaves are **arithmetic $\mathcal{D}$-modules**, but these will mostly lurk in the background.

Aside: there is another form of Drinfeld's lemma involving **perfectoid spaces**. Ongoing work of Fargues–Scholze applies this to the local Langlands correspondence in mixed characteristic $(0, p)$, again with ℓ-adic coefficients for $\ell \neq p$. Work of Carter–KSK–Zábrádi on **multivariate (φ, Γ)-modules** suggests a link with the $\ell = p$ case à la Colmez's p-adic Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$.

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Setup: a formal quotient by Frobenius

Throughout this section...

$X =$ a scheme over $\mathbb{F}_p$
$k =$ an algebraically closed field of characteristic p
$X_k = X \times_{\mathbb{F}_p} k$
$\varphi_k =$ the pullback to X_k of the absolute Frobenius on $\operatorname{Spec} k$

We will consider "X_k/φ_k" as a formal quotient: an object of some type over X_k/φ_k is an object of the same type over X_k equipped with an isomorphism with its φ_k-pullback.

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Coherent sheaves: the original Drinfeld's Lemma

Theorem (Drinfeld, Lau)

For $X/\mathbb{F}_p$ (finite type and) projective, the base extension functor

$$(\text{coherent sheaves on } X) \to (\text{coherent sheaves on } X_k/\varphi_k)$$

is an equivalence of categories and preserves cohomology. (In the latter case, this means the hypercohomology of the complex $\mathcal{E} \stackrel{\varphi_k - 1}{\to} \mathcal{E}$.)

When $X = \operatorname{Spec} \mathbb{F}_p$, this says that a finite-dimensional k-vector space with a semilinear (bijective) φ_k-action has a fixed basis. This is nonabelian Artin–Schreier, a/k/a Katz–Lang (see SGA 7, XXII).

The general case follows using the fact that a coherent sheaf $\mathcal{E}$ on X (resp. X_k) can be recovered from its spaces of sections $H^0(X, \mathcal{E}(n))$ (resp. $H^0(X_k, \mathcal{E}(n))$) for $n \gg 0$.

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Finite étale covers

Let $\mathbf{FEt}(X)$ be the category of finite étale schemes over X.

Corollary

The base extension functor $\mathbf{FEt}(X) \to \mathbf{FEt}(X_k/\varphi_k)$ *is an equivalence.*

This formally reduces to the case where X is affine and of finite type over $\mathbb{F}_p$. In this case, one can choose a projective compactification Y of X; any object of $\mathbf{FEt}(X_k/\varphi_k)$ extends to a finite normal cover of Y_k/φ_k, which by the theorem comes from some finite cover of Y.

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Profinite fundamental groups and lisse sheaves

Corollary

For X connected, X_k/φ_k is connected and for any geometric point $\overline{x} \to X_k$, $\pi_1^{\mathrm{prof}}(X_k/\varphi_k, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X, \overline{x})$.

Warning: in general $\pi_0(X_k) \neq \pi_0(X)$. For example, if $X = \operatorname{Spec} \ell$ is a geometric point, $\pi_0(X_k) \cong \widehat{\mathbb{Z}}$ indexed by identifications of the copies of $\overline{\mathbb{F}}_p$ in k and ℓ; but φ_k acts on $\pi_0(X_k)$ by translation by $\mathbb{Z}$.

Corollary

For $\ell \neq p$ prime, the pullback functor

$$\{\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X\} \to \{\text{lisse } \overline{\mathbb{Q}}_\ell\text{-sheaves on } X_k/\varphi_k\}$$

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Open subschemes and étale sheaves

Corollary

The quasicompact open subschemes of X_k/φ_k are exactly the pullbacks of the quasicompact open subschemes of X.

Quasicompactness lets us reduce to the case where X is affine and of finite type over $\mathbb{F}_p$. In this case, choose a projective compactification Y; given an open subscheme U of X_k invariant under φ_k, apply the theorem to the ideal sheaf on Y_k defining the reduced complement of U.

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For X of finite type over $\mathbb{F}_p$ and $\ell \neq p$ prime, the pullback functor

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Products of two (or more) fundamental groups

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For X_1, X_2 two connected $\mathbb{F}_p$-schemes, one of which is qcqs, put $X = X_1 \times_{\mathbb{F}_p} X_2$ and let $\varphi_1, \varphi_2 : X \to X$ be the partial Frobenius maps. Then X/φ_2 is connected, and for any geometric point $\overline{x} \to X$,

$$\pi_1^{\mathrm{prof}}(X/\varphi_2, \overline{x}) \cong \pi_1^{\mathrm{prof}}(X_1, \overline{x}) \times \pi_2^{\mathrm{prof}}(X_2, \overline{x}).$$

This follows from the case where X_2 is a geometric point using the homotopy exact sequence for a fibration (SGA 1, X).

A similar conclusion holds for n schemes, all but one of which are qcqs.

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Convergent F-isocrystals

Although we eventually want to consider **overconvergent F-isocrystals**, we need to start with a slightly simpler definition.

Let X be a smooth affine scheme over a perfect field k of characteristic p. Fix a formal scheme P smooth over $W(k)$ with $P_k \cong X$ and a lift σ of φ_X to P.

A **convergent F-isocrystal** on X is a finite projective module over $\Gamma(P, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k)[p^{-1}]$-linear connection and a horizontal isomorphism with its σ-pullback.

The resulting $\mathbb{Q}_p$-linear tensor category $\mathbf{F\text{-}Isoc}(X)$ does not depend[†] on P or σ, and extends by glueing to general smooth X. The σ-action then coincides with the functorial φ_X-action.

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Newton polygons

For $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$ and $\overline{x} \to X$ a geometric point lying over $x \in X$, we may pull back $\mathcal{E}$ to $\mathbf{F\text{-}Isoc}(\overline{x})$ and apply the Dieudonné–Manin classification: that pullback decomposes as $\bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$ where for $d = \frac{r}{s} \in \mathbb{Q}$ in lowest terms, $\mathcal{E}_d$ admits a basis killed by $\varphi_X^s - p^r$.

Make the (convex) Newton polygon having slope d with multiplicity $\operatorname{rank} \mathcal{E}_d$ for all d; this depends only on x, not on $\overline{x}$, and is denoted $\mathrm{NP}(\mathcal{E}, x)$.

Theorem (Grothendieck–Katz)

The Newton polygon function $\mathrm{NP}(\mathcal{E}, \bullet)$ *on* $|X|$ *is upper semicontinuous.*

For example, the middle cohomology of the universal elliptic curve over a modular curve has generic Newton slopes $0, 1$ (ordinary), but at isolated points it jumps to $\frac{1}{2}, \frac{1}{2}$ (supersingular).

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Slope filtrations

Theorem (Katz)

If $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant on $|X|$, then $\mathcal{E}$ admits a filtration

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

in which $\mathcal{E}_i/\mathcal{E}_{i-1}$ has all Newton slopes equal to μ_i, and $\mu_1 < \cdots < \mu_l$.

We say that $\mathcal{E}$ is **unit-root** (or **étale**) if $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant on $|X|$ with all slopes equal to 0.

Theorem (Katz, Crew)

The functor $\mathcal{E} \mapsto \mathcal{E}^{\varphi_X}$ defines an equivalence between the (full) category of unit-root objects of $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ and the category of lisse $\mathbb{Q}_p$-sheaves on X.

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If $\mathrm{NP}(\mathcal{E}, \bullet)$ *is constant on* $|X|$*, then* $\mathcal{E}$ *admits a filtration*

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in which $\mathcal{E}_i/\mathcal{E}_{i-1}$ *has all Newton slopes equal to* μ_i*, and* $\mu_1 < \cdots < \mu_l$*.*

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Given $\mathcal{E} \in \mathbf{F}\text{-}\mathbf{Isoc}(X)$, there exists an open dense subscheme U of $\mathcal{E}$ on which $\mathrm{NP}(\mathcal{E}, \bullet)$ is constant. We then obtain a slope filtration in $\mathbf{F}\text{-}\mathbf{Isoc}(U)$. Each successive quotient is, up to a twist[‡], associated to some lisse $\mathbb{Q}_p$-sheaf. However, the extensions between these pieces do not come from lisse sheaves, and so require some analysis directly in $\mathbf{F}\text{-}\mathbf{Isoc}(U)$.

For example, suppose that there are two distinct slopes, so that we have an exact sequence

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The extension class belongs to H^1 of the object $\mathcal{E}_2^\vee \otimes \mathcal{E}_1$, which is not unit-root.

[‡]This twist might be fractional, in which case we must either replace φ_X with a power or extend coefficients from $\mathbb{Q}_p$ to a ramified extension.

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Convergent Φ-isocrystals

For $i = 1, 2$, let X_i be a smooth affine scheme over a perfect field k_i of characteristic p. Fix a formal scheme P_i smooth over $W(k_i)$ with $(P_i)_{k_i} \cong X_i$ and a lift σ_i of φ_{X_i} to P_i.

A **convergent Φ-isocrystal** on $X = X_1 \times_{\mathbb{F}_p} X_2$ is a finite projective module over $\Gamma(P_1 \times_{\mathbb{Z}_p} P_2, \mathcal{O})[p^{-1}]$ equipped with an integrable $W(k_1 \otimes_{\mathbb{F}_p} k_2)[p^{-1}]$-linear connection and commuting horizontal isomorphisms with its σ_i-pullbacks. Let $\Phi\,\mathbf{Isoc}(X)$ be the resulting category; again, this is functorially independent of any choices.

We may make a similar definition for an n-fold product for $n > 2$.

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There is a natural pullback functor from $\mathbf{F}\text{-}\mathbf{Isoc}(X_1)$ to $\Phi\,\mathbf{Isoc}(X)$. When X_2 is a geometric point, this admits the one-sided inverse

$$\mathcal{E} \in \Phi\,\mathbf{Isoc}(X) \mapsto \mathcal{E}^{\varphi_2} \in \mathbf{F}\text{-}\mathbf{Isoc}(X_1)$$

but is **not** an equivalence (see below).

However, for $\mathcal{E}$ as above, the sequence

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If $k_1 = \mathbb{F}_p$, then X is itself smooth over k_2, and an object of $\Phi\,\mathbf{Isoc}(X)$ is just an object of $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ equipped with an isomorphism with its φ_2-pullback.

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*Suppose that X_2 is a geometric point. For $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$, the **total Newton polygon** of $\mathcal{E}$ (i.e., the Newton polygon of the image object in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$), as a function on $|X|$, factors through $|X_1|$.*

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Any irreducible $\mathcal{E} \in \Phi\,\mathbf{Isoc}(X)$ is a subobject of the form $\mathcal{E}_1 \boxtimes \mathcal{E}_2$ for some $\mathcal{E}_i \in \mathbf{F}\text{-}\mathbf{Isoc}(X_i)$.

Again, we first do the case where the total Newton polygon is constant, then use the full faithfulness of restriction.

When $k_1 = k_2 = \mathbb{F}_p$, this can be reformulated as an analogue of the isomorphism

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in terms of Tannakian fundamental groups (Daxin Xu).

A similar statement holds for n-fold products for $n > 2$.

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This can be fixed by considering **overconvergent F-isocrystals**. For X smooth over a perfect field k, the category $\mathbf{F\text{-}Isoc}^\dagger(X)$ can be described by modifying the construction of $\mathbf{F\text{-}Isoc}(X)$ replacing the formal scheme P with a **weak formal scheme**. For instance, if $X = \mathbb{A}^n_k$, then instead of the formal scheme $P = \widehat{\mathbb{A}}^n_{W(k)}$ with

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Arithmetic $\mathcal{D}$-modules

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Given such a "sheaf" on $X_k = X \times_{\mathbb{F}_p} k$ where X is a scheme (or an algebraic stack) of finite type over $\mathbb{F}_p$, there is a maximal open dense subspace U_k on which this object restricts to an overconvergent F-isocrystal. If we start with an object on X_k/φ_k, then (by Drinfeld's lemma) U_k is the base extension of an open subspace U of X.

Is this enough to adapt V. Lafforgue's construction to p-adic coefficients? Time will tell...

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Given such a "sheaf" on $X_k = X \times_{\mathbb{F}_p} k$ where X is a scheme (or an algebraic stack) of finite type over $\mathbb{F}_p$, there is a maximal open dense subspace U_k on which this object restricts to an overconvergent F-isocrystal. If we start with an object on X_k/φ_k, then (by Drinfeld's lemma) U_k is the base extension of an open subspace U of X.

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Isocrystals without Frobenius structure

One can also consider convergent and overconvergent isocrystals without Frobenius structure (although the definition is a bit different).

For $X = X_1 \times_{\mathbb{F}_p} X_2$, are there similar results relating isocrystals on X_1 and X_2 to isocrystals on X equipped with a partial Frobenius action? Our techniques cannot touch this question (even when X_2 is a geometric point).

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Further reading (with links)

Anne T. Carter, KSK, and Gergely Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules, arXiv.

KSK, Sheaves, stacks, and shtukas, Arizona Winter School 2017 lectures.

KSK, Notes on isocrystals, arXiv.

KSK, Étale and crystalline companions, I (arxiv), II (arXiv).

KSK, Several forms of Drinfeld's lemma, RAMpAGe seminar talk.

Eike Lau, On generalised $\mathcal{D}$-shtukas, PhD thesis (Bonn, 2004).

Peter Scholze and Jared Weinstein, *Berkeley Lectures on p-adic Geometry*.