Computing Coleman integrals on modular curves

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joint work (in progress) with Mingjie Chen and Jun Bo Lau

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Coleman’s theory of $p$-adic path integrals

Let $X$ be a curve over $\mathbb{Q}_p$ with good reduction (for simplicity).

Coleman (1985) gave a definition of the path integral $\int_P^Q \omega \in \mathbb{C}_p$ where $\omega$ is a holomorphic differential on $X$ and $P, Q \in X(\mathbb{C}_p)$. (More generally, $\omega$ need only be defined on a generally on a wide open subspace $U$ of the associated rigid analytic space, provided that $P, Q \in U(\mathbb{C}_p)$.)

Among its other uses, Coleman integration plays an important role in effective methods in arithmetic geometry.

- Torsion points (effective Manin-Mumford conjecture).
- Rational points on curves via Chabauty-Coleman method.
- Kim’s nonabelian Chabauty method (specialized to quadratic Chabauty).

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Jacobian arithmetic

The oldest approach to computing Coleman integrals (Wetherell, 1998) used:

- linearity in the endpoints;
- compatibility with the naive definition on a residue disc.

To compute \( \int_D \omega \) for \( D \in \text{Pic}^0(X) \), find a positive integer \( n \) such that \( nD \) projects to zero in the Jacobian of \( X \) over \( \mathbb{F}_p \). One can then write \( nD \) in terms of points in a single residue disc, and then compute
\[
\int_D \omega = n^{-1} \int_{nD} \omega
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by direct integration of power series. (We refer to integrals computed by direct integration in a disc as “tiny integrals”.)

Applying this method in practice depends on Jacobian arithmetic. This is sometimes impractical.
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Frobenius lifts on wide open subspaces

A subsequent approach (Balakrishnan-Bradshaw-K, 2008) used:
- direct computation of tiny integrals;
- change of variables for a Frobenius lift $\varphi$ on a wide open subspace.

Say $\omega$ is a differential form whose class in Monsky-Washnitzer cohomology is a Frobenius eigenvector with eigenvalue $\lambda$; by change of variables,

$$
\lambda \int_P^Q \omega = \int_P^Q \varphi^* \omega + f(Q) - f(P) = \int_{\varphi(P)}^{\varphi(Q)} \omega + f(Q) - f(P)
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where $f$ is an antiderivative of $(\varphi^* - \lambda) \omega$. Now

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\int_P^Q \omega = (\lambda - 1)^{-1} \left( f(Q) - f(P) + \int_P^{\varphi(P)} \omega + \int_{\varphi(Q)}^Q \omega \right)
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where $\lambda - 1 \neq 0$ by weights (i.e., Weil’s proof of RH for curves).

This depends on computing in MW cohomology, and thus on explicit equations. This is sometimes impractical.
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Hecke correspondences on modular curves

We now focus on modular curves. The problem of (provably) finding all rational points on such a curve is of special interest; for example, Mazur’s theorem on torsion points on elliptic curves amounts to finding $X_0(N)(\mathbb{Q})$ for all $N$ (namely, only cusps unless the genus is 0).

Unfortunately, explicit equations for modular curves are often very messy, because these curves are “probably (almost) Brill-Noether general”. (E.g., they are known to have large gonality.)

Let $X$ be a modular curve (e.g., $X_0(N)$). For each prime $\ell$ not dividing $N$, adding $\ell$ to the level gives rise to a new modular curve $X'$ admitting two projections $\pi_1, \pi_2 : X' \to X$ of degree $\ell + 1$.

We may use $X'$ to define the Hecke correspondence $T_\ell$. It acts both on divisors and on differential forms via the same formula:

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We may use $X'$ to define the Hecke correspondence $T_\ell$. It acts both on divisors and on differential forms via the same formula:

$$D \mapsto \pi_2^* \pi_1^* D.$$
Suppose now that \( \omega \) is a Hecke eigenform. Then for any divisor \( D \),

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ap \int_D \omega = \int_{T_p^*(D)} \omega
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where \( a_p \) denotes the eigenvalue of \( T_p \) on \( \omega \). We can rewrite as

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(p + 1 - a_p) \int_D \omega = \int_{(p+1)D - T_p^*(D)} \omega.
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For each closed point \( P \) in \( D \), \((p + 1)P\) and \( T_p^*(P) \) each consist of \( p + 1 \) points in the same residue disc. We can thus compute the right side via tiny integrals; since \( p + 1 - a_p \neq 0 \) (the Ramanujan bound implies \( |a_p| \leq 2\sqrt{p} \)), we can solve for \( \int_D \omega \).
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Avoidance of models

It remains to compute tiny integrals of a Hecke eigenform. If one has access to a model of the curve, one can expand in power series that way.

However, it is not always convenient to compute a model of the curve. Another possible approach is to produce the series expansion over $\mathbb{C}$ using the uniformization by the upper half-plane. The series has coefficients in the eigenvalue field of $\omega$, which is a number field; one can also control the height and denominators of the coefficients. So a sufficiently good complex floating-point approximation (with rigorous error terms) will suffice.

So far we have tested this in some low-genus examples (e.g., $X_0(37)$) by comparing with the model-based method. Asymptotic performance remains to be assessed.
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