p-adic Lafforgue: a road map

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Warning: this talk is highly speculative! In particular, statements asserted without a citation are not written down.

Notation and conventions

$\mathbb{F}_q$	finite field, of characteristic p
X	curve over $\mathbb{F}_q$ (smooth, proj, geom irred)
U	nonempty open subscheme of X
$\mathbb{A}_X$	adele ring of X
K	finite extension of $\mathbb{Q}_p$ with res field $\mathbb{F}_q$
$\mathcal{O}$	ring of integers of K

Lafforgue's work in a nutshell

The Langlands correspondence for function fields: for $\ell \neq p$ prime, there is a bijection between

(a) rank n lisse ℓ-adic sheaves over U ("Galois representations"), and

(b) cuspidal representations of $\mathrm{GL}_n(\mathbb{A}_X)$ unramified at places of U ("automorphic representations"),

which matches eigenvalues of Frobenius at a place $x \in U$ with eigenvalues of the Hecke operator for x (i.e., which matches L-functions).

Goal: do likewise for "p-adic sheaves", by establishing the requisite cohomological machinery (*without* redoing the automorphic side).

Compatible systems (or, Why bother?)

Lafforgue's work implies that every lisse ℓ-adic sheaf on U belongs to a compatible system. Having p-adic Lafforgue would show that:

(a) every "lisse p-adic sheaf" also belongs to a compatible system;

(b) every compatible system includes a "petit camarade cristalline".

It incidentally also serves as a benchmark for measuring current progress on building a p-adic Weil cohomology, which would combine theoretical robustness with practical (numerical) computability. (Another example: one can now recover the Weil conjectures purely within the p-adic framework.)

What are p-adic sheaves?

$\mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger$ = the ring of n-variate power series which converge on some (unspecified) polydisc of radius greater than 1 (i.e., are "overconvergent").

Suppose $A = \mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger/I$ is flat over $\mathcal{O}$ and $U = \mathrm{Spec}(A \otimes_{\mathcal{O}} \mathbb{F}_q)$. There exists a ring homomorphism $\sigma : A \to A$ fixing $\mathcal{O}$ and lifting the q-power Frobenius on U.

Then an *overconvergent F-isocrystal* on U is a finite locally free $A[\frac{1}{p}]$-module M equipped with a connection $\nabla : M \to M \otimes \Omega^{1,\mathrm{cts}}$ and compatible isomorphism $F : \sigma^* M \to M$.

The corresponding cohomology theory, in which this is a coefficient object, is Berthelot's *rigid cohomology* (modeled on de Rham cohomology).

Lisse vs. constructible (or, Caveat auditor)

We do not yet have a good p-adic analogue of constructible sheaves! These should arise from an analogue of algebraic $\mathcal{D}$-modules, but the analogue of holonomicity remains mysterious. (It is not even known to be stable under inverse image along a closed immersion!)

In practice, it seems we can get around this. E.g., one expects that for any $f : X \to Y$ and any overconvergent F-isocrystal $\mathcal{E}$ on X, there exist higher direct images

$$R^i f_* \mathcal{E}, \qquad R^i f_! \mathcal{E}$$

on an open dense subscheme U of Y. Moreover, if f is smooth proper, one should be able to take $U = Y$ (conjecture of Berthelot).

These are not known in full, but may now be tractable (as in finiteness of rigid cohomology).

First steps

- Chebotarev density: If $\mathcal{E}, \mathcal{F}$ are overconvergent F-isocrystals on U with the same charpoly of Frobenius at (almost) every point, then they are isomorphic. Apparently easy: follows from Chebotarev in the convergent category, plus full faithfulness of the functor "overconvergent-to-convergent".

- Define local epsilon factors. This is currently being done by Marmora; may resemble Beilinson-Bloch-Esnault's analogue for complex local systems using irregularities.

- Verify the product formula for epsilon factors. Probably not hard: follow Laumon, using Huyghe's p-adic Fourier transform.

Fixed point formulas

One needs a Lefschetz-Verdier trace formula: given $f : Z \to Z$ with $Z/\mathbb{F}_q$ proper, the sum

$$\sum_{i=0}^{2\dim(Z)} (-1)^i \mathrm{Trace}(f, H^i(Z))$$

is equal to a sum of (usually mysterious) local contributions concentrated on the fixed subscheme of f. (Similarly also with coefficients.)

This is easily established for Frobenius, which induces a trace class operator *on the chain level*. Moreover, composing any given f with a high power of Frobenius gives a trace class operator. (Analogue: Fujiwara's proof of Deligne's conjecture that composing with a high power of Frobenius "trivializes" local contributions.)

For general maps (or correspondences), the construction by Petrequin of cycle classes may help; not sure if this is relevant here.

Lafforgue's class of "serene" stacks

An algebraic (Artin) stack $\mathcal{X}$ over a scheme S is *serene* if:

(S1) The morphism $\mathcal{X} \to S$ is separated.

(S2) The morphism $\mathcal{X} \to S$ is of finite type.

(S3) Each point of $\mathcal{X}$ admits a Zariski open neighborhood $\mathcal{V}$ admitting a finite flat cover $V \to \mathcal{V}$ by an algebraic space such that $\mathtt{Aut}(V/\mathcal{V})$ acts transitively on reduced geometric fibres.

In particular, the diagonal morphism

$$\Delta_{\mathcal{X}/S} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$$

is (representable and) finite. Lafforgue suggests that maybe one can get by with this instead of S3.

The big desiderata (ouch)

Construct rigid cohomology for serene stacks (over $\mathbb{F}_q$, or ultimately $U \times U$) and verify:

1. finiteness of cohomology of local systems;
2. Künneth decomposition and Poincaré duality;
3. Lefschetz-Verdier and analogue of Fujiwara.

Good warmup: do these for Deligne-Mumford stacks. (Even easier warmup: do these for quotients of a scheme by action of a finite group.)

Final random musing

The serene stacks in Lafforgue are moduli spaces of "shtukas". A shtuka (of rank r) on an $\mathbb{F}_q$-scheme S is a vector bundle $\mathcal{E}$ (of rank r) over $X \times S$ equipped with:

- a diagram

$$\mathcal{E} \overset{j}{\hookrightarrow} \mathcal{E}' \overset{t}{\hookleftarrow} \mathcal{E}''$$

  where $\mathcal{E}', \mathcal{E}''$ are vector bundles of rank r over $X \times S$ and j, t are injections whose cokernels are invertible over $\mathcal{O}_S$ and are supported on sections $\infty, 0 : S \to X$;

- a "complete homomorphism"

$$(\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathcal{E} \Rightarrow \mathcal{E}$$

  (further details omitted).

Is there a useful crystalline analogue?