Frobenius structures on hypergeometric equations

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These slides can be downloaded from https://kskedlaya.org/slides/.

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I acknowledge that my workplace sits on unceded ancestral land of the Kumeyaay Nation.

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Hypergeometric differential operators

For n a positive integer and

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_n), \underline{\beta} = (\beta_1, \dots, \beta_n) \in \mathbb{Q}^n,$$

the **hypergeometric differential operator** with parameters $\underline{\alpha}, \underline{\beta}$ is the differential operator in one variable z given by

$$P(\underline{\alpha}; \underline{\beta})(D) := z \prod_{i=1}^{n} (D + \alpha_i) - \prod_{j=1}^{n} (D + \beta_j - 1), \qquad D := z \frac{d}{dz}.$$

When $\beta_n = 1$ and $\alpha_i, \beta_j \notin \mathbb{Z}_{\leq 0}$, it admits as a formal solution the **(Clausen–Thomae) hypergeometric series**

$${}_nF_{n-1} \left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \beta_1, \dots, \beta_{n-1} \end{matrix} \middle| z \right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k} \frac{z^k}{k!} \in \mathbb{Q}[\![z]\!]$$

where $(\alpha)_k$ means the **(rising) Pochhammer symbol**

$$(\alpha)_k := \alpha(\alpha + 1) \cdots (\alpha + k - 1).$$

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The effect of repeated indices

Recall that

$$P(\underline{\alpha};\underline{\beta})(D) = z\prod_{i=1}^{n}(D+\alpha_i) - \prod_{j=1}^{n}(D+\beta_j-1), \qquad D := z\frac{d}{dz}.$$

One may check directly that for $\delta \in \mathbb{Q}$,

$$(D+\delta-1)P(\underline{\alpha};\underline{\beta}) = P(\underline{\alpha},\delta;\underline{\beta},\delta)$$
$$P(\underline{\alpha};\underline{\beta})(D+\delta) = P(\underline{\alpha},\delta;\underline{\beta},\delta+1).$$

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Recall that

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$$P(\underline{\alpha}; \underline{\beta})(D + \delta) = P(\underline{\alpha}, \delta; \underline{\beta}, \delta + 1).$$

Since $P(\underline{\alpha}, \underline{\beta})$ is invariant under permutation within $\underline{\alpha}$ or $\underline{\beta}$, it follows that

$$P(\underline{\alpha}; \underline{\beta})(D + \alpha_i - 1) = (D + \alpha_i - 1)P(\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_m; \underline{\beta})$$
$$(D + \beta_j - 1)P(\underline{\alpha}; \underline{\beta}) = P(\underline{\alpha}; \beta_1, \dots, \beta_j - 1, \dots, \beta_n)(D + \beta_j).$$

In classical terminology, we have identified **intertwining operators***
between $P(\underline{\alpha}, \underline{\beta})$ and the operators obtained by performing an integer shift on any parameter.

*More precisely, these are intertwining operators for the associated monodromy representations, provided that $\alpha_i \neq \beta_j \pmod{\mathbb{Z}}$ for any i, j.

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Hypergeometric systems

In terms of the hypergeometric differential operator written as

$$P(\underline{\alpha};\underline{\beta})(D) = (z-1)(D^n + a_{n-1}D^{n-1} + \cdots + a_0 D),$$

we obtain a linear differential operator on length-n column vectors:

$$N + D, \qquad N := \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & -1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}.$$

The solutions of the equation $(N+D)(\mathbf{v}) = 0$ are of the form

$$\mathbf{v} = \begin{pmatrix} y \\ D(y) \\ \vdots \\ D^{n-1}(y) \end{pmatrix} \quad \text{where} \quad P(\underline{\alpha};\underline{\beta})(D)(y) = 0.$$

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Hypergeometric connections

A linear differential operator of the form $N + D$ in turn defines a rank-n vector bundle $\mathcal{E}_{\underline{\alpha},\underline{\beta}}$ equipped with an integrable logarithmic connection $\nabla_{\underline{\alpha},\underline{\beta}}$. This connection is irreducible as long as

$$\alpha_i \not\equiv \beta_j \pmod{\mathbb{Z}} \qquad (i, j = 1, \ldots, n),$$

which we assume hereafter.

The intertwining operators induce **meromorphic** isomorphisms

$$(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}}) \cong (\mathcal{E}_{\underline{\alpha}',\underline{\beta}'}, \nabla_{\underline{\alpha}',\underline{\beta}'})$$

whenever

$$\underline{\alpha}' \equiv \underline{\alpha}, \underline{\beta}' \equiv \underline{\beta} \pmod{\mathbb{Z}}.$$

That is, the meromorphic isomorphism class of $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$ is invariant under the natural action of $(\mathbb{Z}^n \times \mathbb{Z}^n) \rtimes (S_n \times S_n)$ on $\mathbb{Q}^n \times \mathbb{Q}^n$.

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Exponents of hypergeometric connections

The connection $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$ is singular only at $z = 0, 1, \infty$, where it has the following residual eigenvalues (a/k/a **exponents**):

$$\begin{aligned} z = 0: &\qquad 1-\beta_1, \dots, 1-\beta_n \\ z = \infty: &\qquad \alpha_1, \dots, \alpha_n \\ z = 1: &\qquad 0, \dots, n-2, \gamma, \qquad \gamma := \sum_{i=1}^n \beta_i - \sum_{i=1}^n \alpha_i. \end{aligned}$$

The residue matrices at 0 and ∞ have minimal polynomials

$$(T-1+\beta_1)\cdots(T-1+\beta_n), \qquad (T-\alpha_1)\cdots(T-\alpha_n).$$

These properties **uniquely** characterize the connection $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$.

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Frobenius lifts do not exist in general. E.g., if P is the formal completion of a smooth projective curve of genus ≥ 2 over $\mathbb{Z}_p$, then P does not admit a Frobenius lift.

However, Frobenius lifts do exist if P is affine. For example, given any formally étale map $P \to \widehat{\mathbb{A}}^m_{\mathbb{Z}_p}$, the Frobenius map $t_i \mapsto t_i^p$ on $\widehat{\mathbb{A}}^m_{\mathbb{Z}_p}$ lifts uniquely to P.

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Frobenius pullback

Let $\mathcal{E}$ be a vector bundle on the Raynaud generic fiber $P_{\mathbb{Q}_p}$ equipped with an integrable connection. Under certain conditions, there are **canonical** natural transformations $\sigma_1^*\mathcal{E} \cong \sigma_2^*\mathcal{E}$ for any two Frobenius lifts σ_1, σ_2, defined using Taylor series. For example, this holds when $\mathcal{E}$ is a convergent isocrystal.[†]

When this occurs, we may interpret the various functors σ^* as a single functor Φ_p^*, the **algebraic Frobenius pullback**. We may also extend Φ_p^* to cases where P does not admit a Frobenius lift.

This remains true if we allow logarithmic connections with respect to a relative strict normal crossings divisor on P.

If P is the completion of a smooth proper $\mathbb{Z}_p$-scheme X, then by rigid GAGA we may interpret both $\mathcal{E}$ and $\Phi_p^*\mathcal{E}$ as connections on $X_{\mathbb{Q}_p}$.

[†] For $p > 2$, it also holds when the connection arises by base extension from $\mathbb{Z}_p$ to $\mathbb{Q}_p$.

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Algebraic Frobenius structures

An **algebraic Frobenius structure** is an isomorphism $\Phi_p^* \mathcal{E} \cong \mathcal{E}$. The existence of such forces $\mathcal{E}$ to be an isocrystal.

Such a structure always exists if $\mathcal{E}$ is "geometric" (i.e., appears in the relative rigid cohomology of some smooth proper morphism over P_k).

If P is the completion of a smooth proper $\mathbb{Z}_p$-scheme X, then by rigid GAGA an algebraic Frobenius structure induces an isomorphism of connections on $X_{\mathbb{Q}_p}$. **However**, this hides the fact that the construction of the functor Φ_p^* is not itself algebraic!

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Hypergeometric algebraic Frobenius structures

For $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}_{(p)}^n$, the series ${}_nF_{n-1}\left(\begin{smallmatrix}\alpha_1,\dots,\alpha_n \\ \beta_1,\dots,\beta_{n-1}\end{smallmatrix} \middle| z\right)$ converges p-adically for $|z| < 1$. This implies that the base extension of $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$ from $\mathbb{Q}$ to $\mathbb{Q}_p$ is a convergent log-isocrystal.

Using the rigidity of hypergeometric connections, we can prove:

Theorem

The base extension of $(\mathcal{E}_{p\underline{\alpha},p\underline{\beta}}, \nabla_{p\underline{\alpha},p\underline{\beta}})$ from $\mathbb{Q}$ to $\mathbb{Q}_p$ is isomorphic to the algebraic Frobenius pullback of $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$.

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Balanced parameters

We say that $\underline{\alpha} \in \mathbb{Q}^n$ is **balanced** if for any positive integer s, the quantity

$$\#\{i \in \{1, \dots, n\} : \alpha_i \equiv \tfrac{r}{s} \pmod{\mathbb{Z}}\}$$

is the same for all $r \in \mathbb{Z}$ coprime to s. For example,

$$(\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{4}) \text{ is balanced but } (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{4}) \text{ is not.}$$

If $\underline{\alpha}, \underline{\beta}$ are balanced, then $(\underline{\alpha}, \underline{\beta})$ is $(\mathbb{Z}^n \times \mathbb{Z}^n) \rtimes (S_n \times S_n)$-equivalent to $(p\underline{\alpha}, p\underline{\beta})$ for any prime p for which $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}^n_{(p)}$.

Corollary

Suppose that $\underline{\alpha}, \underline{\beta}$ are balanced. Then for every p for which $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}^n_{(p)}$, the isomorphism $(\mathcal{E}_{p\underline{\alpha}, p\underline{\beta}}, \nabla_{p\underline{\alpha}, p\underline{\beta}}) \cong (\mathcal{E}_{\underline{\alpha}, \underline{\beta}}, \nabla_{\underline{\alpha}, \underline{\beta}})$ induces an algebraic Frobenius structure on $(\mathcal{E}_{\underline{\alpha}, \underline{\beta}}, \nabla_{\underline{\alpha}, \underline{\beta}})$ over $\mathbb{Q}_p$.

Balanced parameters

We say that $\underline{\alpha} \in \mathbb{Q}^n$ is **balanced** if for any positive integer s, the quantity

$$\#\{i \in \{1, \dots, n\} : \alpha_i \equiv \tfrac{r}{s} \pmod{\mathbb{Z}}\}$$

is the same for all $r \in \mathbb{Z}$ coprime to s. For example,

$$(\tfrac{1}{4}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{4}) \text{ is balanced but } (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{4}) \text{ is not.}$$

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Contents

A reformulation of the hypergeometric series

Assume that $\alpha_i, \beta_j \notin \mathbb{Z}_{\geq 0}$. We previously defined

$$ {}_nF_{n-1}\left(\begin{matrix} \alpha_1, \dots, \alpha_n \\ \beta_1, \dots, \beta_{n-1} \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k}{(\beta_1)_k \cdots (\beta_{n-1})_k} \frac{z^k}{k!}. $$

If $\beta_n = 1$, this can be rewritten as

$$ \frac{\Gamma(\underline{\beta})}{\Gamma(\underline{\alpha})} \sum_{k=0}^{\infty} \frac{\Gamma(\underline{\alpha} + k)}{\Gamma(\underline{\beta} + k)} z^k, $$

writing $\Gamma(\underline{\alpha}) := \Gamma(\alpha_1) \cdots \Gamma(\alpha_n)$ and $\underline{\alpha} + k := (\alpha_1 + k, \dots, \alpha_n + k)$.

Using the identity $\Gamma(x)\Gamma(1-x) = \frac{x}{\sin(\pi x)}$, this may be further rewritten as

$$ \sum_{k=0}^{\infty} \left(\prod_{i=1}^{n} \frac{\Gamma(\alpha_i + k)\Gamma(1 - \beta_i - k)}{\Gamma(\alpha_i)\Gamma(1 - \beta_i)} \right) ((-1)^n z)^k. $$

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The function Γ can be interpreted as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt.$$

That is, we are integrating a multiplicative character of $\mathbb{C}$ against an additive character; for $z \in \mathbb{Z}$, we can instead think of these as characters of $\mathbb{C}/2\pi\mathbb{Z} \cong \mathbb{C}^\times$.

Fix a finite field $\mathbb{F}_q$ and a nontrivial additive character $\psi_q : \mathbb{F}_q \to \mathbb{C}$. For each multiplicative character $\chi : \mathbb{F}_q^\times \to \mathbb{C}^\times$, define the **Gauss sum**

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The Gross–Koblitz(–Boyarsky) formula expresses $g(\chi)$ in terms of the Morita p-adic Gamma function Γ_p. One can then compute $H_q(\underline{\alpha}, \underline{\beta}|t)$ via a comparable formula (Cohen–Rodriguez Villegas–Watkins):

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Assume hereafter that $\underline{\alpha}, \underline{\beta}$ are balanced.

Beukers–Cohen–Mellit describe an explicit morphism to $\mathbb{P}^1_{\mathbb{Q}}$ for which $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$ occurs as a Gauss–Manin connection, and count points on fibers in terms of $H_q(\underline{\alpha}, \underline{\beta}|t)$. This can be reinterpreted as follows.

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For $\underline{\alpha}, \underline{\beta}$ balanced and p prime such that $\underline{\alpha}, \underline{\beta} \in \mathbb{Z}^n_{(p)}$, there is an algebraic Frobenius structure on $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}}) \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that for every power q of p and every $t \in \mathbb{F}_q \setminus \{1\}$, the trace of q-power Frobenius at t equals $H_q(\underline{\alpha}, \underline{\beta}|t)$.

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Applications of the (p-adic) BCM formula

The p-adic reformulation of the BCM formula has been implemented in MAGMA (Watkins) and then ported to SAGE (Chapoton, K, Roe). It is very efficient!

Better yet, one expects to implement an "average polynomial time" strategy to, for fixed $t \in \mathbb{Q}$, compute $H_q(\underline{\alpha}, \underline{\beta}|\overline{t})$ for all[§] prime powers $q \leq X$ in time[¶] $O(X^{1+\epsilon})$. So far this is implemented for computing $H_p(\underline{\alpha}, \underline{\beta}|\overline{t}) \pmod{p}$ and seems to be quite practical up to say $X = 2^{32}$ (Costa–K–Roe).

This could then be used to build extensive tables of motivic L-functions appearing in hypergeometric families. This is desired for LMFDB.

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Algebraic Frobenius structures and the BCM formula(?)

Question: is there an interpretation/new proof of the BCM formula in terms of the algebraic Frobenius structure on $(\mathcal{E}_{\underline{\alpha},\underline{\beta}}, \nabla_{\underline{\alpha},\underline{\beta}})$?

One mild reinterpretation is to view the original point-counting proof through the lens of Dwork cohomology.

However, I am rather looking for an answer that also provides a q-analogue of the BCM formula, in the sense of q-hypergeometric series of Aomoto etc. Warmup question: is there a reasonable‖ q-deformation of the theory of Gauss sums?

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Assume now that $\mathcal{E}$ is geometric** over $\mathbb{Q}$. Then for some N, for each $p \in M_N$ we have an algebraic Frobenius structure

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Let X be a smooth proper $\mathbb{Q}$-scheme. Let Z be a normal crossings divisor on X. Let $\mathcal{E}$ be a vector bundle on X equipped with a logarithmic (along Z) integrable connection.

For N a positive integer, let M_N be the multiplicative monoid of integers coprime to N.

Assume now that $\mathcal{E}$ is geometric** over $\mathbb{Q}$. Then for some N, for each $p \in M_N$ we have an algebraic Frobenius structure

$$F_p : \Phi_p^*(\mathcal{E} \times_{\mathbb{Q}} \mathbb{Q}_p) \cong \mathcal{E} \times_{\mathbb{Q}} \mathbb{Q}_p$$

arising from crystalline realizations.

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The question

For some N, can we find a family of connections $\mathcal{E}_m$ on X indexed by $m \in M_N$ and a family of isomorphisms $\mathcal{E}_m \cong \mathcal{E}$ such that for each $m \in M_n$ and each prime $p \in M_N$, there is an isomorphism completing the diagram

$$\begin{array}{ccc} \mathcal{E}_{mp} \times_{\mathbb{Q}} \mathbb{Q}_p & \dashrightarrow & \Phi_p^*(\mathcal{E}_m \times_{\mathbb{Q}} \mathbb{Q}_p) \\ & \searrow \quad \swarrow_{F_p} & \\ & \mathcal{E}_m \times_{\mathbb{Q}} \mathbb{Q}_p & \end{array}$$

in which the left diagonal is induced by $\mathcal{E}_{mp} \cong \mathcal{E} \cong \mathcal{E}_m$ and the right diagonal arises from $F_p : \Phi_p^*(\mathcal{E} \times_{\mathbb{Q}} \mathbb{Q}_p) \cong \mathcal{E} \times_{\mathbb{Q}} \mathbb{Q}_p$ via the isomorphism $\mathcal{E}_m \cong \mathcal{E}$?

Content of the question

The question is nontrivial in two aspects.

- It is not clear from the construction that $\Phi_p^*(\mathcal{E} \times_{\mathbb{Q}} \mathbb{Q}_p)$ descends to a connection $\mathcal{E}_p$ on X (not on $X \times_{\mathbb{Q}} \mathbb{Q}_p$). The definition depends crucially on convergence of some p-adic limits.
- For $p_1, p_2 \in M_N$ prime, we have a commuting diagram

$$\begin{array}{ccc} \mathcal{E}_{mp_1p_2} & \longrightarrow & \mathcal{E}_{mp_1} \\ \downarrow & & \downarrow \\ \mathcal{E}_{mp_2} & \longrightarrow & \mathcal{E}_m \end{array}$$

  in which the vertical (resp. horizontal) arrows can be interpreted in terms of F_{p_1} (resp. F_{p_2}). But this interpretation requires base extension to $\mathbb{Q}_{p_1}$ or $\mathbb{Q}_{p_2}$, whereas commutativity of the diagram only makes sense over $\mathbb{Q}$.

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Examples from hypergeometric connections

Our previous theorem asserts that such a structure always exists for balanced hypergeometric connections, taking N to be the least common denominator of $\underline{\alpha} \cup \underline{\beta}$.

One can also formulate a similar result for unbalanced hypergeometric connections, at the expense of replacing the base field $\mathbb{Q}$ with $\mathbb{Q}(\mu_N)$ for some suitable N.

Question: is this an instance of a more general phenomenon? And if so, does this have anything to do with **global** arithmetic L-functions?

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Further directions

- **GKZ hypergeometric systems**: these live over higher-dimensional toric varieties and should be an easy adaptation.
- **Shimura varieties**: here one has only a weaker form of rigidity, so one probably has to use special ("CM") points to pin things down. On the other hand, one can handle some cases where the connection is not yet known to be geometric (after Esnault–Groechenig, Diu–Lan–Liu–Zhu, Klevdal–Patrikis...).
- q**-de Rham cohomology**: building on ideas of Aomoto, Pridham, Masullo, and Bhatt–Scholze, one eventually hopes to prove some general results.
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