Toric coordinates in relative p-adic Hodge theory

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Toric geometry and applications
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The field of p-adic numbers

Throughout this talk, p will be a fixed prime number, and $\mathbb{Q}_p$ will be the *field of p-adic numbers*. We will think of $\mathbb{Q}_p$ in two different ways.

- As $\mathbb{Z}_p[\frac{1}{p}]$, where $\mathbb{Z}_p$ (the *ring of p-adic integers*) is the completion of the ring $\mathbb{Z}$ with respect to the ideal (p). That is,

$$\mathbb{Z}_p = \varprojlim \left(\cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \right).$$

- As the completion of $\mathbb{Q}$ for the *p-adic absolute value*: for $e, r, s \in \mathbb{Z}$ with r, s not divisible by p,

$$\left| p^e \frac{r}{s} \right|_p = p^{-e}.$$

The second description gives rise to several notions of *analytic geometry* over $\mathbb{Q}_p$. Throughout most of today's talks, the relevant version will be that of Berkovich; more on that later.

What is Hodge theory?

Hodge theory begins with the study of the relationship between different cohomology theories for algebraic varieties over $\mathbb{C}$, such as Betti (singular) cohomology and (algebraic or holomorphic) de Rham cohomology. These define "the same" vector spaces over $\mathbb{C}$, but come naturally with different extra structure: an integral lattice on Betti cohomology, a Hodge filtration on de Rham cohomology.

One is thus led to introduce *Hodge structures* axiomatizing this setup (and *variations of Hodge structures*), and to study them on their own right. One obtains some information that one can transfer back to algebraic geometry.

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What is p-adic Hodge theory?

p-adic Hodge theory begins with a corresponding study with $\mathbb{C}$ replaced by a finite extension K of $\mathbb{Q}_p$. The most interesting cohomology theories on a variety X are now étale cohomology of $X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})$ with $\mathbb{Q}_p$-coefficients, and algebraic de Rham cohomology.

But things are more complicated! The two sets of cohomology groups only become "the same" after extending scalars to a large topological $\mathbb{Q}_p$-algebra $\mathbf{B}_{\mathrm{dR}}$ introduced by Fontaine. Also, the two sets of extra structures (Galois action on étale cohomology, Hodge filtration and crystalline Frobenius on de Rham cohomology) can be reconstructed from each other.

So let's focus on continuous representations of the absolute Galois group G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces, such as those from étale cohomology. Surprisingly, these are susceptible to methods of positive characteristic (like Artin-Schreier theory); more on this shortly.

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Applications of p-adic Hodge theory

Having been introduced in the early 1980s, use of p-adic Hodge theory in arithmetic geometry has exploded in the last decade.

For example, recent work on *modularity of Galois representations* (part of the Langlands program), extending the resolution of the Fermat problem by Wiles, depends crucially on p-adic Hodge theory. Key results include the proofs of Serre's modularity conjecture (Khare-Wintenberger) and the Sato-Tate conjecture (Taylor et al.). The latter says that for E an elliptic curve over $\mathbb{Q}$, the average distribution of

$$\frac{p + 1 - \#E(\mathbb{F}_p)}{\sqrt{p}}$$

over all primes p is always the semicircular distribution on $[-2, 2]$.

What is relative p-adic Hodge theory?

In arithmetic geometry, G_K also occurs the fundamental group for the étale topology on either Spec(K) or the associated analytic space over $\mathbb{Q}_p$. One is thus led to study continuous representations (still on finite dimensional $\mathbb{Q}_p$-vector spaces) of étale fundamental groups of analytic spaces over $\mathbb{Q}_p$. Again this has to do with comparison between étale and de Rham cohomology, but now for a smooth proper morphism of analytic spaces.

In this lecture, we'll focus on one key aspect, the description of the *étale topology* of an analytic space over $\mathbb{Q}_p$ in terms of positive characteristic geometry. This makes crucial use of *local toric coordinates*.

This transfer from mixed to positive characteristic has some applications outside of p-adic Hodge theory. We'll discuss two at the end, but surely others exist!

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The field of norms equivalence

Let $\mathbb{Q}_p(\mu_{p^\infty})$ be the field obtained from $\mathbb{Q}_p$ by adjoining all p-power roots of unity, and fix a surjection $\mathbb{Q}_p[\mathbb{Q}_p/\mathbb{Z}_p] \twoheadrightarrow \mathbb{Q}_p(\mu_{p^\infty})$. (One might call this choice a *p-adic orientation* by analogy with the situation over $\mathbb{R}$.)

The *field of norms* construction of Fontaine-Wintenberger defines an equivalence of the Galois theories of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p((\pi))$, and hence an isomorphism of Galois groups. Here, it will be more convenient to express this result as an equivalence of tensor categories

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p(\mu_{p^\infty})) \cong \mathbf{F\acute{E}t}(\mathbb{F}_p((\pi))),$$

where $\mathbf{F\acute{E}t}(\bullet)$ denote the category of finite étale algebras over a ring $\bullet$. (Over a field, these are just direct sums of finite separable field extensions.)

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Deep ramification

The existence of the field of norms equivalence is explained by the fact that $\mathbb{Q}_p(\mu_{p^\infty})$ is a highly ramified extension of $\mathbb{Q}_p$ (that is, the integral closure of $\mathbb{Z}_p$ is far from being a formally étale $\mathbb{Z}_p$-algebra).

The original proof of Fontaine-Wintenberger relied on careful analysis of higher ramification. Recent proofs rely instead of these easier facts.

- The Frobenius endomorphism F of $\mathbb{Z}_p[\mu_{p^\infty}]/(p)$ is surjective.
- The inverse limit

$$\varprojlim \left(\cdots \xrightarrow{F} \mathbb{Z}_p[\mu_{p^\infty}]/(p) \xrightarrow{F} \mathbb{Z}_p[\mu_{p^\infty}]/(p) \right)$$

  is isomorphic to the π-adic completion of $\mathbb{F}_p[\![\pi]\!][\pi^{p^{-\infty}}]$, with $(\cdots, \zeta_p - 1, \zeta_1 - 1)$ corresponding to π.

This observation will be needed later in order to generalize the Fontaine-Wintenberger theorem.

Galois descent

For $K \in \mathbf{F\acute{E}t}(\mathbb{Q}_p)$, the action of $\mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p) \cong \mathbb{Z}_p^\times$ on $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{p^\infty})$ transfers to the corresponding $A \in \mathbf{F\acute{E}t}(\mathbb{F}_p((\pi)))$ so that

$$\gamma(1+\pi) = (1+\pi)^\gamma = \sum_{n=0}^{\infty} \binom{\gamma}{n} \pi^n \qquad (\gamma \in \mathbb{Z}_p^\times).$$

Galois theory *below* $\mathbb{Q}_p(\mu_{p^\infty})$ does not transfer to positive characteristic: any nontrivial subgroup of $\mathbb{Z}_p^\times$ acts on $\mathbb{F}_p((\pi))$ only fixes $\mathbb{F}_p$.

Instead, we remain *above* $\mathbb{Q}_p(\mu_{p^\infty})$. Using Galois descent and the field of norms, we obtain equivalences

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \mathbb{Z}_p^\times\text{-}\mathbf{F\acute{E}t}(\mathbb{Q}_p(\mu_{p^\infty})) \cong \mathbb{Z}_p^\times\text{-}\mathbf{F\acute{E}t}(\mathbb{F}_p((\pi))),$$

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Analytic fields

An *analytic field* is a field K which is complete with respect to a multiplicative nonarchimedean norm $|\bullet|$. Write $v(\bullet) = -\log|\bullet|$ for the corresponding valuation.

For example, $\mathbb{Q}_p$ is a analytic field for the p-adic norm $|p| = p^{-1}$, and $\mathbb{F}_p((\pi))$ is an analytic field for the π-adic norm $|\pi| = p^{-p/(p-1)}$.

Any finite extension of an analytic field can again be viewed as an analytic field. Consequently, any infinite algebraic extension of an analytic field can be completed to obtain an analytic field. (If we start with an algebraic closure, the resulting analytic field is also algebraically closed.)

Banach rings and their spectra

A *Banach ring* is a ring A complete with respect to a submultiplicative norm $|\cdot|$. The *Gel'fand spectrum* $\mathcal{M}(A)$ of A is the subset of $[0, +\infty)^A$ consisting of multiplicative seminorms α dominated by the given norm. (*Multiplicative* means $\alpha(xy) = \alpha(x)\alpha(y)$. *Seminorm* means $\alpha(x) = 0$ does not imply $x = 0$. *Dominated* means $\alpha(x) \leq c|x|$ for some $c = c(\alpha)$.)

Under the product topology (a/k/a the *Berkovich topology*), this set is nonempty and compact; moreover, the supremum over the spectrum computes the *spectral seminorm*

$$|x|_{\mathrm{sp}} = \lim_{n\to\infty} |x^n|^{1/n}.$$

We say A is *uniform* when $|\bullet|_{\mathrm{sp}} = |\bullet|$ (i.e., $|x^2| = |x|^2$ for all $x \in A$).

If $A = C_0(X)$ for a compact topological space X, then A is uniform and $\mathcal{M}(A) \cong X$. But for A nonarchimedean, $\mathcal{M}(A)$ can be surprisingly large!

The constructible topology

Let A be a Banach ring. A *rational subspace* of $\mathcal{M}(A)$ is a subset

$$\{\alpha \in \mathcal{M}(A) : \alpha(f_1) \leq p_1\alpha(g), \ldots, \alpha(f_m) \leq p_m\alpha(g)\}$$

for some $f_1, \ldots, f_m, g \in A$ which generate the unit ideal and some $p_1, \ldots, p_m > 0$.

The *constructible topology* on $\mathcal{M}(A)$ is the one generated by rational subspaces. It is finer than the Berkovich topology.

Both the Berkovich and constructible topologies have natural interpretations in terms of tropical geometry of toric varieties. More on that later.

Affinoid algebras

For K an analytic field and $r_1, \dots, r_n > 0$, let $K\{T_1/r_1, \dots, T_n/r_n\}$ be the completion of $K[T_1, \dots, T_n]$ for the Gauss norm

$$\left| \sum_{i_1,\dots,i_n} c_{i_1,\dots,i_n} T_1^{i_1} \cdots T_n^{i_n} \right| = \max\{|c_{i_1,\dots,i_n}| r_1^{i_1} \cdots r_n^{i_n}\}.$$

Any quotient of a $K\{T_1/r_1, \dots, T_n/r_n\}$ is called an *affinoid algebra* over K in the sense of Berkovich. (In classical rigid analytic geometry, one requires $r_1 = \cdots = r_n = 1$; Berkovich calls these *strictly affinoid algebras*.)

We will mostly consider only *reduced* affinoid algebras over K. Any such A carries a distinguished uniform norm, given by taking the quotient norm for a surjection $K\{T_1/r_1, \dots, T_n/r_n\} \twoheadrightarrow A$, then passing to the spectral norm.

Analytic spaces over K are glued (using the constructible topology) spectra of affinoid algebras. But spectra of other Banach rings are useful too!

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Tropicalization of an affine toric variety

Let K be an analytic field. View the valuation v on K as a homomorphism from the multiplicative monoid of K to the monoid $\mathbf{R} = \mathbb{R} \cup \{+\infty\}$.

Let σ be a strictly rational convex polyhedral cone. Let $\mathrm{Spec}(K[S_\sigma])$ be the associated affine toric variety over K.

Put $\mathbf{Trop}(\sigma) = \mathrm{Hom}(S_\sigma, \mathbf{R})$; this space carries a natural rational polyhedral structure. In particular, it admits a natural topology as well as a *constructible topology* generated by rational polyhedral subsets.

Let $K(\sigma)$ be the analytification of $\mathrm{Spec}(K[S_\sigma])$. It admits a natural evaluation map $e_\sigma : K(\sigma) \to \mathbf{Trop}(\sigma)$.

Toric frames and nonarchimedean geometry

For A a reduced affinoid algebra over K, a *toric frame* of A is an unramified morphism $\psi : \mathcal{M}(A) \to K(\sigma_\psi)$ of analytic spaces for some σ_ψ. This includes any composition of locally closed immersions and finite étale covers. (One could get away without étale covers, except that we want to talk about the étale topology later.)

For each ψ, we get an evaluation map

$$e_\psi : \mathcal{M}(A) \to K(\sigma_\psi) \to \mathbf{Trop}(\sigma_\psi).$$

If we view the collection of toric frames for ψ as an inverse system (where transition maps are induced by maps of affine toric varieties), the resulting map $e : \mathcal{M}(A) \to \varprojlim_\psi \mathbf{Trop}(\sigma_\psi)$ is a homeomorphism for both the natural and constructible topologies (Payne).

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The basic idea (after Faltings)

From now on, take $K = \mathbb{Q}_p$.

Recall that the equivalence $\mathbf{F\acute{E}t}(\mathbb{Q}_p(\mu_{p^\infty})) \cong \mathbf{F\acute{E}t}(\mathbb{F}_p((\pi)))$ relied on the fact that Frobenius is surjective on $\mathbb{Z}_p[\mu_{p^\infty}]/(p)$. To describe finite étale algebras over a reduced affinoid algebra A over $\mathbb{Q}_p$, we would similarly like to pass to a larger algebra where Frobenius is surjective on the ring of integral elements (those of norm at most 1) modulo p.

This is where toric varieties come in! They are in a sense the only natural class of schemes over $\mathbb{Z}_p$ admitting lifts of Frobenius (induced by the p-th power map on the underlying monoid).

Deeply ramified covers from toric frames

Let $\psi : \mathcal{M}(A) \to \mathbb{Q}_p(\sigma_\psi)$ be a toric frame. For $n = 0, 1, \ldots$, let $A_{\psi,n}$ be the reduced quotient of $A_\psi[\mu_{p^n}, S_\sigma^{p^{-n}}]$, viewed as an affinoid algebra carrying its uniform norm. We may take the completed direct limit to form $A_{\psi,\infty}$ (which is not an affinoid algebra, only a Banach ring).

Theorem (K-Liu, Scholze)

For $A_{\psi,\infty}^+ = \{x \in A_{\psi,\infty} : |x| \leq 1\}$, the Frobenius endomorphism F on $A_{\psi,\infty}^+/(p)$ is surjective.

This is easy when ψ is an immersion. One picks up the general case in the course of describing finite étale $A_{\psi,\infty}$-algebras; more on this shortly.

Topology in mixed and positive characteristic

Put $\overline{A}_\psi^+ = \varprojlim(\cdots \xrightarrow{F} A_{\psi,\infty}^+ \xrightarrow{F} A_{\psi,\infty}^+)$; this is a ring of characteristic p. Put $\overline{A}_\psi = \overline{A}_\psi^+[\pi^{-1}]$ for $\pi = (\dots, \zeta_p - 1, \zeta_1 - 1)$.

Theorem (K-Liu, Scholze)

There is a natural (in A and ψ) bijection $\mathcal{M}(A_{\psi,\infty}) \cong \mathcal{M}(\overline{A}_\psi)$ which is a homeomorphism for both the Berkovich and constructible topologies.

This is already nontrivial even when ψ is an immersion.

One proof is to describe $A_{\psi,\infty}$ in terms of Witt vectors over $\overline{A}_\psi$ and use the fact that the Teichmüller map behaves a bit like a ring homomorphism (even though it is only multiplicative).

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Étale covers

Theorem (K-Liu, Scholze)

There is a natural (in A and ψ) equivalence of categories

$$\mathbf{F\acute{E}t}(A_{\psi,\infty}) \cong \mathbf{F\acute{E}t}(\overline{A}_{\psi})$$

provided that one equips both $A_{\psi,\infty}$ and $\overline{A}_{\psi}$ with the toric log structure.

By working locally for the Berkovich topology, this reduces to the following generalization of Fontaine-Wintenberger. (The critical case is to lift a ramified Artin-Schreier extension of L'.)

Theorem (K-Liu, Scholze)

Let L be an analytic field of characteristic 0, not discretely valued, such that the Frobenius F on $L^+/(p)$ is surjective. Put $(L')^+ = \varprojlim_F L^+/(p)$ and $L' = \mathrm{Frac}(L')^+$. Then L' is a perfect analytic field of characteristic p, and there is a natural (in L) equivalence $\mathbf{F\acute{E}t}(L) \cong \mathbf{F\acute{E}t}(L')$.

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Galois descent

Let H be the group of homomorphisms from S_σ to the additive monoid $\mathbb{Z}_p$. The equivalence $\mathbf{F\acute{E}t}(A_{\psi,\infty}) \cong \mathbf{F\acute{E}t}(\overline{A}_\psi)$ is compatible with the action of $\Gamma = \mathbb{Z}_p^\times \ltimes H$ on $A_{\psi,\infty}$ (and the induced one on $\overline{A}_\psi$) generated by the Galois action on $\mathbb{Q}_p(\mu_{p^\infty})$ and maps of the form

$$s^{1/p^n} \mapsto \zeta_{p^n}^{\lambda(s)} s^{1/p^n} \qquad (s \in S_\sigma, n \in \mathbb{Z}_{\geq 0})$$

for $\lambda \in H$.

One can thus describe $\mathbf{F\acute{E}t}(A)$ using finite étale algebras over $\overline{A}_\psi$ with Γ-action. However, the Γ-action must be required to be continuous (this is automatic for $A = \mathbb{Q}_p$).

Also, $\mathcal{M}(A) = \mathcal{M}(A_{\psi,\infty})/H$, so one can describe the whole étale topology on $\mathcal{M}(A)$ in terms of $\overline{A}_\psi$.

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Generalized (ϕ, Γ)-modules (K, Liu)

As in ordinary *p*-adic Hodge theory, one can describe étale local systems in $\mathbb{Q}_p$-vector spaces on $\mathcal{M}(A)$ using modules over $W(\overline{A}_\psi)$. One can also pass to other *p-adic period rings* which are convenient for other constructions; this generalizes results of Fontaine, Cherbonnier-Colmez, Berger, etc.

This construction is closely related to Scholze's cosntruction of the relative comparison isomorphism.

The weight-monodromy conjecture (Scholze)

Deligne's *weight-monodromy conjecture* concerns the comparison of two filtrations on the ℓ-adic étale cohomology of a smooth proper $\mathbb{Q}_p$-variety.

Scholze has proved some new cases of WMC, e.g., for a smooth complete intersection in a complete toric variety over a finite extension of $\mathbb{Q}_p$. This proceeds by globalizing the previous construction to transfer a neighborhood of the complete intersection into positive characteristic, then using Deligne's proof of the analogue of WMC over $\mathbb{F}_p((\pi))$.

The direct summand conjecture (Bhatt)

The *direct summand conjecture* in commutative algebra (Hochster) asserts that for R a regular ring and $R \to S$ a module-finite ring homomorphism, the exact sequence

$$0 \to R \to S \to S/R \to 0$$

of R-modules is always split. This is known when R contains a field.

One can use the constructions described here to treat some new cases of DSC for R of mixed characteristic, e.g., when $S[1/p]$ has toroidal singularities.