Frobenius structures on hypergeometric equations

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Contents

Hypergeometric equations and hypergeometric series

Let n be a positive integer. For $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n), \underline{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$, we consider the hypergeometric equation (for $D = z\frac{d}{dz}$)

$$(z(D + \alpha_1) \cdots (D + \alpha_n) - (D + \beta_1 - 1) \cdots (D + \beta_n - 1))(y) = 0.$$

This equation is regular with the following singularities and exponents:

$$\begin{aligned} z = 0 : &\qquad 1 - \beta_1, \ldots, 1 - \beta_n \\ z = \infty : &\qquad \alpha_1, \ldots, \alpha_n \\ z = 1 : &\qquad 0, \ldots, n-2, \gamma, \qquad \gamma := \sum \beta_i - \sum \alpha_i. \end{aligned}$$

The monodromy representation can be described explicitly (see Beukers–Heckman); it is irreducible provided that $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j.

There are intertwining operators for integer shifts of the parameters; we may thus normalize all parameters to have real part in $[0, 1)$.

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Solutions of hypergeometric equations

Define the rising Pochhammer symbol

$$(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1).$$

If $\beta_i \notin \{0,-1,-2,\dots\}$ for $i = 1,\dots,n-1$, then the hypergeometric series

$${}_nF_{n-1}\left(\begin{matrix}\alpha_1,\dots,\alpha_n\\ \beta_1,\dots,\beta_{n-1}\end{matrix}\middle|\, z\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k\cdots(\alpha_n)_k}{(\beta_1)_k\cdots(\beta_{n-1})_k}\frac{z^k}{k!}$$

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Formal solutions of a hypergeometric equation

Suppose that for some $i \in \{1, \dots, n\}$, $\beta_j - \beta_i \notin \mathbb{Z}$ for all $j \neq i$. Then one has a formal solution of the hypergeometric equation given by

$$z^{1-\beta_i} {}_nF_{n-1} \left(\begin{matrix} \alpha_1 - \beta_i + 1, \dots, \alpha_n - \beta_i + 1 \\ \beta_1 - \beta_i + 1, \dots, \widehat{\beta_i - \beta_i + 1}, \dots, \beta_n - \beta_i + 1 \end{matrix} \middle| z \right).$$

If $\beta_j - \beta_i \notin \mathbb{Z}$ for all $j \neq i$, these expressions constitute a formal solution basis at $z = 0$. If in addition $\underline{\beta} \in \mathbb{Q}^n$, these form a genuine $\mathbb{C}$-basis of the solutions in the Puiseux field $\bigcup_{m=1}^{\infty} \mathbb{C}((z^{1/m}))$.

If $\beta_j - \beta_i \in \mathbb{Z}$ for some i, j, one can obtain a formal solution basis by differentiating with respect to parameters; when $\underline{\beta} \in \mathbb{Q}^n$, these live in $\bigcup_{m=1}^{\infty} \mathbb{C}((z^{1/m}))[\log z]$. For simplicity, I will (mostly) omit this case.

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Differential systems

For the purposes of considering Frobenius structures, it is convenient to work with first-order differential systems. For a system of the form $N\mathbf{v} + D(\mathbf{v}) = 0$ where N is the companion matrix

$$N := \begin{pmatrix} 0 & -1 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 0 & -1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix},$$

the solutions are the vectors of the form

$$\mathbf{v} = \begin{pmatrix} y \\ D(y) \\ \vdots \\ D^{n-1}(y) \end{pmatrix} \quad \text{where} \quad D^n(y) + a_{n-1}D^{n-1}(y) + \cdots + a_0 y = 0.$$

Frobenius structures

Fix a prime p. Let K be the completion of $\mathbb{Q}_p(z)$ for the Gauss norm, viewed as a differential field for the derivation $D = z\frac{d}{dz}$.

Let $\sigma : K \to K$ be a *Frobenius lift*, i.e., a continuous endomorphism satisfying $|\sigma(z) - z^p| < 1$. Define the quantity

$$c_\sigma := \frac{D(\sigma(z))}{\sigma(z)};$$

it satisfies $|c_\sigma| < 1$.

Given a differential system defined by a matrix N over K, a *Frobenius structure* with respect to σ is given by a matrix F satisfying

$$NF + D(F) = c_\sigma F \sigma(N).$$

In the language of connections, the map $\mathbf{v} \mapsto F\sigma(\mathbf{v})$ defines an isomorphism of the pullback connection (via σ) with the original one.

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Change of the Frobenius lift

In principle, the definition of a Frobenius structure depends on the choice of σ. However, it turns out that this is illusory.

Define the sequence of matrices

$$N_0 = 1, \qquad N_{k+1} = (N - k + 1)N_k + D(N_k) \quad (k = 0, 1, \dots).$$

Then for any other Frobenius lift σ', the formula

$$F' = \sum_{n=0}^{\infty} \frac{(\sigma'(z) - \sigma(z))^n}{n!} F\sigma(N_k)$$

converges and defines a Frobenius lift with respect to σ'. This can be used to transfer some information between different choices of σ.

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Convergence of local solutions

In general, the Cauchy theorem does not apply to p-adic power series: the exponential series satisfies a nonsingular differential equation on the entire z-line, but has a finite radius of convergence.

However, consider a differential system with no singularities in the disc $|z| < 1$. Then the existence of a Frobenius structure implies (by an argument of Dwork) that the formal solutions in $\mathbb{Q}_p[\![z]\!]^n$ converge on the disc $|z| < 1$.

A similar argument applies in the case of a single regular singularity in the disc, located at $z = 0$, with exponents in $\mathbb{Q} \cap \mathbb{Z}_p$. Note that in this case, the existence of a Frobenius structure implies that the exponents form a multisubset of $\mathbb{Q}/\mathbb{Z}$ which is stable under multiplication by p.

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Frobenius structures on hypergeometric equations

We say that parameters $\underline{\alpha}, \underline{\beta} \in \mathbb{Q}^n$ are *Galois-stable* if the multisets $\{e^{2\pi i \alpha_j} : j = 1, \dots, n\}$, $\{e^{2\pi i \beta_j} : j = 1, \dots, n\}$ are Galois-stable. That is, any two classes in $\mathbb{Q}/\mathbb{Z}$ of the same order occur with equal multiplicities.

Theorem (Dwork)

If $\underline{\alpha}, \underline{\beta} \in \mathbb{Q}^n$ are Galois-stable, then the differential system associated to the hypergeometric equation admits a Frobenius structure. If in addition $\underline{\alpha}, \underline{\beta}$ are disjoint modulo $\mathbb{Z}$, the Frobenius structure is unique up to a $\mathbb{Q}_p$-scalar multiple.

Without the Galois-stable condition, one gets a matrix F for which $N'F + D(F) = c_\sigma F \sigma(N)$, where N' is the companion matrix for the hypergeometric equation with parameters $\underline{\alpha}' = p\underline{\alpha} \mod \mathbb{Z}$, $\underline{\beta}' = p\underline{\beta} \mod \mathbb{Z}$. These matrices have some good p-adic analyticity properties.

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Let X be a smooth proper variety over a number field K. For $i = 0, \ldots, 2\dim(X)$, one can form an (incomplete) L-function

$$L_{X,i}(s) = \prod_{\mathfrak{p}} \det(1 - \mathrm{Norm}(\mathfrak{p})^{-s}\mathrm{Frob}_{\mathfrak{p}}, H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{I_\mathfrak{p}})^{-1},$$

where $\mathfrak{p}$ runs over prime ideals of the integer ring $\mathfrak{o}_K$; this is a Dirichlet series which converges absolutely for $\mathrm{Re}(s) \gg 0$. (The determinant is nominally a polynomial in $\mathrm{Norm}(\mathfrak{p})^{-s}$ with coefficients in $\mathbb{Q}_\ell$, but in fact the coefficients belong to $\mathbb{Q}$.)

Conjecturally, after completing with suitable Γ-factors, one gets a function which admits meromorphic continuation to $\mathbb{C}$ and a functional equation with respect to $s \mapsto i + 1 - s$. When this is known it is often very deep (e.g., for elliptic curves over totally real fields).

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L-functions of motives

A *motive*[1] of weight i over K gives rise to, for some smooth proper X/K, a linear projector on $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for each ℓ which are "induced by a uniform geometric construction." For M such an object, we may define its L-function $L_M(s)$ by analogy with $L_{X,i}(s)$; we then have

$$L_{X,i}(s) = L_M(s)L_{M'}(s)$$

where M' is the complementary motive (i.e., the family of complementary projectors). This generalizes the factorization of the Dedekind zeta function of a number field into Artin L-functions.

One similarly defines morphisms between motives, which may go between different varieties. In the resulting category, one has isomorphic motives with different ambient varieties, e.g., the full 1-motives of isogenous elliptic curves or abelian varieties.

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Motives and differential equations

One can similarly define a *family of motives* over $\mathbb{Q}(z)$. In this case, one also obtains (using the de Rham realization) a connection on $\mathbb{Q}(z)$, the *Gauss–Manin connection* of the family; if the latter is expressed as the differential system associated to an equation, the latter is called a *Picard–Fuchs equation* of the family.

This equation will admits a Frobenius structure at p for almost all p (which must be normalized suitably). For $z \in \overline{\mathbb{Q}}$ at which the equation is nonsingular, the specializations at z can be used[2] to compute the L-function of the specialized motive.

Example: for the Legendre family of elliptic curves $y^2 = x(x-1)(x-z)$, the Gaussian hypergeometric equation (i.e., $n = 2$, $\underline{\alpha} = (1/2, 1/2)$, $\underline{\beta} = (1, 1)$) appears as a Picard–Fuchs equation. The Frobenius structure in this case was constructed explicitly by Dwork.

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Hypergeometric motives

For any parameters $\underline{\alpha}, \underline{\beta} \in \mathbb{Q}^n$ which are Galois-stable and disjoint modulo $\mathbb{Z}$, the hypergeometric equation arises from a family of motives described by Katz. These are of interest in arithmetic geometry for several reasons.

- There are good algorithms for computing the associated L-functions, including the Cohen–Rodriguez Villegas–Watkins p-adic adaptation of the Beukers–Cohen–Mellit trace formula; this is implemented in `Magma` and Sage. (See later in this lecture for an alternative.)
- The motives that occur include various interesting examples, including some motives associated to K3 surfaces, Calabi–Yau threefolds, etc.
- They also include some more exotic examples which are hard to replicate in other ways. (More precisely: there is an algorithm to identify hypergeometric motives with particular Hodge numbers.)
- Putting this together, we obtain a mechanism for testing conjectures about the L-functions of motives, particularly questions about special values (conjectures of Beilinson, Deligne, Bloch–Kato, etc.).

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- There are good algorithms for computing the associated L-functions, including the Cohen–Rodriguez Villegas–Watkins p-adic adaptation of the Beukers–Cohen–Mellit trace formula; this is implemented in `Magma` and `Sage`. (See later in this lecture for an alternative.)
- The motives that occur include various interesting examples, including some motives associated to K3 surfaces, Calabi–Yau threefolds, etc.
- They also include some more exotic examples which are hard to replicate in other ways. (More precisely: there is an algorithm to identify hypergeometric motives with particular Hodge numbers.)
- Putting this together, we obtain a mechanism for testing conjectures about the L-functions of motives, particularly questions about special values (conjectures of Beilinson, Deligne, Bloch–Kato, etc.).

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Hypergeometric motives and the LMFDB

The LMFDB (L-Functions and Modular Forms Database) is an ongoing collaborative project to build a "star chart" of L-functions and arithmetic-geometric objects associated with them:

- number fields (Jones–Roberts tables);
- elliptic curves over $\mathbb{Q}$ and some other number fields (Cremona tables);
- some curves of genus > 1 (Sutherland tables);
- classical and some higher-rank modular forms (Hilbert, Bianchi, Siegel);
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This is work in progress; some steps need to be rigorously justified. However, it sems to work in practice; see `this Jupyter notebook on CoCalc` for a demonstration in Sage that I gave at the Hausdorff Institute in March, in which I compare results against Sage's implementation of the trace formula of Cohen et al.

Similar strategies have been used in previous algorithms, originating with the work of Lauder. A particularly good implementation, in the context of families of smooth projective hypersurfaces, has been produced by Pancratz–Tuitman.

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Changes of basis

The effect of changing basis on a differential system, and a Frobenius structure, is as follows:

$$N \mapsto U^{-1}NU + U^{-1}D(U), \qquad F \mapsto U^{-1}F\sigma(U).$$

Over $\mathbb{Q}_p[\![z]\!]$, we may write down an invertible matrix U for which

$$N_0 := U^{-1}NU + U^{-1}D(U) = \mathrm{Diag}(\beta_1 - 1, \dots, \beta_n - 1):$$

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$$y_j = {}_nF_{n-1}\left(\left. \begin{matrix} \alpha_1 - \beta_j + 1, \dots, \alpha_n - \beta_j + 1 \\ \beta_1 - \beta_j + 1, \dots, \widehat{\beta_j - \beta_j + 1}, \dots, \beta_n - \beta_j + 1 \end{matrix} \right| z \right).$$

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The commutation relation between N_0 and $F_0 := U^{-1} F \sigma(U)$ now reads

$$N_0 F_0 + D(F_0) = p F_0 \sigma(N_0).$$

It implies that $(F_0)_{ij} = 0$ unless $\beta_i \equiv p\beta_j \pmod{\mathbb{Z}}$. In the latter case, for some $\lambda_i \in \mathbb{Q}_p^\times$ we have

$$(F_0)_{ij} = \lambda_i z^{p\beta_j - \beta_i + 1}.$$

Suppose that we somehow knew how to compute the λ_i. We could then use these as an *initial condition* to compute F_0, and hence $F = U F_0 \sigma(U^{-1})$, as a matrix over $\mathbb{Q}_p[\![z]\!]$.

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A conjecture for the initial condition

Recall that the previous strategy depends on identifying the scalars λ_i appearing in the transformed Frobenius matrix F_0; this is where the normalization comes into the story.

Conjecture

For i, j with $\beta_i \equiv p\beta_j \pmod{\mathbb{Z}}$, we have

$$\lambda_i = p^{Z(\beta_j) - \min_k\{Z(\beta_k)\}}(-1)^{1+Z(\beta_i)} \prod_{k=1}^{n} \frac{\Gamma_p((\alpha_k - \beta_i) \mod 1)/\Gamma_p(\alpha_k)}{\Gamma_p((\beta_k - \beta_i) \mod 1)/\Gamma_p(\beta_k)},$$

where Z denotes the "zigzag function" associated to the parameters:

$$Z(x) := \#\{k : \alpha_k \leqslant x\} - \#\{k : \beta_k \leqslant x\}.$$

This has been tested for thousands of random parameters with $n \leqslant 6$. It *may* follow from work in Dwork's *Generalized Hypergeometric Functions*.

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This function appears in the *Gross–Koblitz formula* expressing Gauss sums in terms of the values of Γ_p at rational numbers. The appearances of Γ_p in the expression for λ_i is surely related!

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Explicit bounds on the Frobenius structure: p-adic direction

Recall that F has entries in the completion of $\mathbb{Q}_p[z, z^{-1}, (z-1)^{-1}]$ within K; we can thus only hope to compute a p-adic approximation to the entries of F, and hence a p-adic approximation to the characteristic polynomial of F. Using the Weil conjectures, it is easy to predict how accurate the latter approximation needs to be in order to provably recover the L-function.

To translate this into an estimate for the required precision for the approximation of F, we need to bound the p-adic norms of the entries of F. Some (limited) experimentation suggests the following.

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The p-adic norms of the entries of F are bounded by p^k for

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In particular, if $\alpha_i \not\equiv \beta_j \pmod{p}$ for all i, j, then F has entries in $\mathfrak{o}_K$.

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It is possible (although we have not yet done so) to obtain such bounds by comparing the Frobenius structures with respect to σ and $\sigma' : z \mapsto (z-1)^p + 1$. However, experiments suggest that the optimal bounds are stronger than what a direct approach would give; this may be connected to *supercongruences* of finite hypergeometric sums.

With this bound in hand, one can then go back and control the working p-adic precision needed for the computation of U. For $p \gg 0$, the number of terms of the power series needed will be $O(p) \ll p^2$, so the requisite truncation of $p^n U$ will have entries in $\mathbb{Z}_p$. Moreover, the requisite truncation of $\sigma(U)^{-1}$ will have entries in $\mathbb{Z}_p$, with no rescaling required!

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Postscript: average polynomial time methods

As a side benefit of this analysis, it should be possible to adapt the previous computation so that, for any hypergeometric motive over $\mathbb{Q}$, the factors of the L-functions at all primes $p \leqslant X$ (omitting primes of bad reduction) are computed in *average polynomial time* per p; this has previously been achieved in the context of hyperelliptic curves by Harvey (and implemented by Harvey–Sutherland in genus $\leqslant 3$). The key idea is contained in the following related result.

Theorem (Costa–Gerbicz–Harvey)

There is an algorithm which computes $\left(\frac{p-1}{2}\right)! \pmod{p^2}$ for all odd primes $p \leqslant X$ in time $O(X \log^m X)$ for some m. That is, the average time per p is polynomial in $\log p$.

The idea: one needs to compute $(\lfloor X/2 \rfloor)!$ modulo all p for $X/2 \leqslant p \leqslant X$; so we do it modulo the product instead, using fast multiplication in $\mathbb{Z}$.

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Theorem (Costa–Gerbicz–Harvey)

There is an algorithm which computes $\left(\frac{p-1}{2}\right)! \pmod{p^2}$ *for all odd primes* $p \leqslant X$ *in time* $O(X \log^m X)$ *for some* m*. That is, the average time per* p *is polynomial in* $\log p$*.*

The idea: one needs to compute $(\lfloor X/2 \rfloor)!$ modulo all p for $X/2 \leqslant p \leqslant X$; so we do it modulo the product instead, using fast multiplication in $\mathbb{Z}$.

Contents

Conclusion

To summarize:

- hypergeometric equations whose parameters are Galois-stable, distinct modulo $\mathbb{Z}$, and p-adically integral admit Frobenius structures;
- when $\underline{\beta}$ has no repeats[3] modulo $\mathbb{Z}$, these are effectively computable as power series around $z = 0$ given an initial condition[4];
- and with enough concrete analysis of p-adic and z-adic precision, this becomes an effective algorithm for computing the L-functions of hypergeometric motives.

If time permits, we can look at my demonstration from March. Otherwise, thank you for your attention.

[3] It should be possible to extend to the case of repeated parameters by working not over $\mathbb{Q}_p$, but some nilpotent deformation thereof.

[4] In most other contexts, the initial condition is obtained as a nonsingular point, as this avoids some touchy issues regarding p-adic nearby cycles.

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