# Towards uniformity over $p$ in $p$-adic Hodge theory

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Théorie de Hodge $p$-adique,
équations différentielles $p$-adiques et leurs applications
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# The field of norms equivalence

Let $p$ be a prime number, and fix an identification

$$\mathbb{Q}_p(\mu_{p^\infty}) \cong \mathbb{Q}_p[\mathbb{Q}_p/\mathbb{Z}_p]/\left(\sum_{i=0}^{p-1}\left\{\frac{i}{p}\right\}\right);$$

i.e., fix a coherent sequence of $p$-power roots of unity in $\mathbb{Q}_p(\mu_{p^\infty})$.

Let $\mathbf{FÉt}(\bullet)$ denote the category of finite étale algebras over a ring $\bullet$. By Fontaine-Wintenberger, there is a distinguished equivalence of categories

$$\mathbf{FÉt}(\mathbb{Q}_p(\mu_{p^\infty})) \cong \mathbf{FÉt}(\mathbb{F}_p((\overline{\pi}))).$$

The action of $\mathbb{Z}_p^\times$ on $\mathbb{Q}_p/\mathbb{Z}_p$ corresponds to the action on $\mathbb{F}_p((\overline{\pi}))$ via

$$\gamma : 1+\overline{\pi} \mapsto (1+\overline{\pi})^\gamma = \sum_{n=0}^{\infty}\binom{\gamma}{n}\overline{\pi}^n \qquad (\gamma \in \mathbb{Z}_p^\times).$$

# The goal

By Fontaine-Wintenberger plus Galois descent, we obtain an equivalence

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \{\text{finite étale } \mathbb{F}_p((\overline{\pi}))\text{-algebras with } \mathbb{Z}_p^\times\text{-action}\}$$

in which the prime $p$ is referenced in numerous ways. Our goal is to restructure the right side of this equivalence so that $p$ intervenes *only* via the choice of a $p$-adic absolute value on $\mathbb{Q}$.

# Motivation: the $p$-adic Langlands correspondence

Recall that one can interpret continuous $p$-adic Galois representations of $G_{\mathbb{Q}_p}$ in terms of $(\varphi, \Gamma)$-*modules*. These in turn give rise to representations of the group

$$\begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$$

as in Colmez's analysis of the $p$-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. (Colmez's next step is to reinterpret in terms of distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ via the Amice transform; we won't get that far here.)

# Motivation: the Bost-Connes system

Meanwhile, Bost and Connes observed that a suitably completed Hecke algebra for the groups

$$\begin{pmatrix} \mathbb{Z}^\times & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \subseteq \begin{pmatrix} \mathbb{Q}^\times & \mathbb{Q} \\ 0 & 1 \end{pmatrix}$$

is a $C^*$-algebra with a one-parameter family of automorphisms (*time evolution*) from which the Riemann zeta function arises naturally as a partition function in the context of quantum statistical mechanics. The group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times$ also appears naturally as a symmetry group.

# Motivation: coefficients on the BC-system

This formal parallel suggests that the $C^*$-algebra in the Bost-Connes system might admit coefficients corresponding to motives. By cultivating the analogy with $p$-adic Hodge theory further, we will see that one should use a larger $C^*$-algebra instead.

If one could really define a cohomology theory on motives over $\mathrm{Spec}(\mathbb{Z})$ with values in a category of modules over such a $C^*$-algebra, one could then try to use this to give a spectral interpretation of $L$-functions. But we won't even begin to discuss this today.

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# Overview

As a first step, we reframe the field of norms equivalence in a manner that does not seem to achieve much towards our goal of eliminating $p$: we continue to work with fields of characteristic $p$ and their Frobenius automorphisms, and we also introduce $p$-typical Witt vectors. However, this will prepare us for later.

As a bonus, the final equivalence we write down can be established directly, and one can then read the slides in reverse order to recover a proof of the Fontaine-Wintenberger theorem. This proof, which involves no higher ramification theory, arises in the work of K-Liu on relative $p$-adic Hodge theory. One finds a related proof in Scholze's study of *perfectoid spaces*, using Gabber-Ramero's analysis of almost étale extensions of valuation rings.

# Adding $\mathbb{Z}_p^\times$

We begin with the equivalence stated previously:

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \{\text{finite étale } \mathbb{F}_p((\overline{\pi}))\text{-algebras with } \mathbb{Z}_p^\times\text{-action}\}.$$

We will make a series of modifications on the right side of this equivalence, indicated by changing green text to red text.

To begin with, the action of $\mathbb{Z}_p^\times$ on finite étale $\mathbb{F}_p((\overline{\pi}))$-algebras is always continuous (that is, the action map $\mathbb{Z}_p^\times \times \bullet \to \bullet$ is continuous). We thus obtain

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \mathbb{F}_p((\overline{\pi}))\text{-algebras} \\ \text{with arbitrary } \mathbb{Z}_p^\times\text{-action} \end{matrix} \right\}.$$

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# Completion in positive characteristic

Let $\mathbf{E}_{\mathbb{Q}_p}$ denote the completed perfect closure of $\mathbb{F}_p((\overline{\pi}))$, normalized with $|\overline{\pi}| = p^{-p/(p-1)}$. Then

$$\mathbf{F\acute{E}t}(\mathbb{F}_p((\overline{\pi}))) \cong \mathbf{F\acute{E}t}(\mathbb{F}_p((\overline{\pi}))^{\mathrm{perf}})$$

(e.g., because radicial morphisms are universal homeomorphisms) and

$$\mathbf{F\acute{E}t}(\mathbb{F}_p((\overline{\pi}))^{\mathrm{perf}}) \cong \mathbf{F\acute{E}t}(\mathbf{E}_{\mathbb{Q}_p})$$

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# Adding $p^{\mathbb{Z}}$

Any finite étale $\mathbf{E}_{\mathbb{Q}_p}$-algebra admits a unique bijective extension of the Frobenius $\varphi$. We may thus extend the action of $\mathbb{Z}_p^\times$ to an action of $\mathbb{Q}_p^\times$ with $p$ acting via the extension of $\varphi$. The natural topology on $\mathbb{Q}_p^\times$ is the product topology on $p^{\mathbb{Z}} \cdot \mathbb{Z}_p^\times$ with the discrete topology on the first factor, so we get

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \mathbf{E}_{\mathbb{Q}_p}\text{-algebras with} \\ \text{continuous } \mathbb{Z}_p^\times\text{-action} \end{matrix} \right\}.$$

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# Lifting to $p$-typical Witt vectors

Let $W_p(\mathbf{E}_{\mathbb{Q}_p})$ be the ring of $p$-typical Witt vectors over $\mathbf{E}_{\mathbb{Q}_p}$. It is a Cohen ring, so base extension defines an equivalence

$$\mathbf{F\acute{E}t}(W_p(\mathbf{E}_{\mathbb{Q}_p})) \cong \mathbf{F\acute{E}t}(\mathbf{E}_{\mathbb{Q}_p}).$$

Equip $W_p(\mathbf{E}_{\mathbb{Q}_p})$ with the *weak topology*: a sequence $\{\sum_{n=0}^{\infty} p^n[\overline{x}_{n,m}]\}_{m=0}^{\infty}$ converges if and only if $\{\overline{x}_{n,m}\}_{m=0}^{\infty}$ converges in $\mathbf{E}_{\mathbb{Q}_p}$ for each $n$. We then have

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \mathbf{E}_{\mathbb{Q}_p}\text{-algebras with} \\ \text{continuous } \mathbb{Q}_p^{\times}\text{-action} \end{matrix} \right\}.$$

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$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } W_p(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

# A formal inverse limit

On $W_p(\mathbf{E}_{\mathbb{Q}_p})$, $\varphi$ coincides with the Witt vector Frobenius $F_p$. To interpret $\varphi^{-1}$ similarly, use the bijectivity of $\varphi$ to identify $W_p(\mathbf{E}_{\mathbb{Q}_p})$ with

$$\varprojlim W_p(\mathbf{E}_{\mathbb{Q}_p}) = \varprojlim \left( \cdots \stackrel{F_p}{\to} W_p(\mathbf{E}_{\mathbb{Q}_p}) \stackrel{F_p}{\to} W_p(\mathbf{E}_{\mathbb{Q}_p}) \right);$$

the latter formally admits an inverse of $F_p$ (given by left shifting) which coincides with $\varphi^{-1}$. We then have

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } W_p(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

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$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \varprojlim W_p(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

# Overconvergence

Define $|\cdot|_W : W_p(\mathbf{E}_{\mathbb{Q}_p}) \to [0, +\infty]$ by

$$\left|\sum p^n[\overline{x}_n]\right|_W = \sup_n\{|\overline{x}_n|\}.$$

Let $W_p^0(\mathbf{E}_{\mathbb{Q}_p})$ be the subset on which $|\cdot|_W$ is finite; this is a ring complete under the multiplicative norm $|\cdot|_W$.

For $r > 0$, complete $W_p^0(\mathbf{E}_{\mathbb{Q}_p}) \otimes_{\mathbb{Z}} \mathbb{Z}$ for the product seminorm $|\cdot|_r = |\cdot|_W^r \otimes |\cdot|_p$ to obtain $W_p^r(\mathbf{E}_{\mathbb{Q}_p})$. That is,

$$W_p^r(\mathbf{E}_{\mathbb{Q}_p}) = \left\{\sum p^n[\overline{x}_n] \in W_p(\mathbf{E}_{\mathbb{Q}_p}) : \lim_{n\to\infty} p^{-n}|\overline{x}_n|^r = 0\right\}$$

and $|\sum p^n[\overline{x}_n]|_r = \sup_n\{p^{-n}|\overline{x}_n|^r\}$.

Equip the henselian ring $W_p^\dagger(\mathbf{E}_{\mathbb{Q}_p}) = \bigcup_{r>0} W_p^r(\mathbf{E}_{\mathbb{Q}_p})$ with the *LF topology*: a sequence converges if and only if it converges under some $|\cdot|_r$.

# Overconvergence and inverse limits

For $r > 0$, put

$$\underleftarrow{W}_p^r(\mathbf{E}_{\mathbb{Q}_p}) = \varprojlim \left( \cdots \stackrel{F_p}{\to} W_p^{rp}(\mathbf{E}_{\mathbb{Q}_p}) \stackrel{F_p}{\to} W_p^r(\mathbf{E}_{\mathbb{Q}_p}) \right);$$

the natural map $\underleftarrow{W}_p^r(\mathbf{E}_{\mathbb{Q}_p}) \to W_p^r(\mathbf{E}_{\mathbb{Q}_p})$ is a bijection. Consequently,

$$\underleftarrow{W}_p^\dagger(\mathbf{E}_{\mathbb{Q}_p}) = \bigcup_{r>0} \underleftarrow{W}_p^r(\mathbf{E}_{\mathbb{Q}_p}) \to W_p^\dagger(\mathbf{E}_{\mathbb{Q}_p})$$

is also a bijection, giving

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \underleftarrow{W}_p(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

# Overconvergence and inverse limits

For $r > 0$, put

$$\underleftarrow{W}^r_p(\mathbf{E}_{\mathbb{Q}_p}) = \varprojlim \left( \cdots \xrightarrow{F_p} W^{rp}_p(\mathbf{E}_{\mathbb{Q}_p}) \xrightarrow{F_p} W^r_p(\mathbf{E}_{\mathbb{Q}_p}) \right);$$

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# Inverse limits and the theta map

Identify $\mathbf{E}^+_{\mathbb{Q}_p} = \{x \in \mathbf{E}_{\mathbb{Q}_p} : |x| \leq 1\}$ with the inverse limit

$$\mathbf{E}^+_{\mathbb{Q}_p} \cong \varprojlim \mathbb{Z}_p[\mu_{p^\infty}]/(p), \qquad \overline{\pi} \mapsto (\dots, \overline{\zeta}_p - 1, \overline{\zeta}_1 - 1).$$

There is a multiplicative map $\overline{\theta} : \mathbf{E}^+_{\mathbb{Q}_p} \to \widehat{\mathbb{Z}_p[\mu_{p^\infty}]}$ taking $\overline{x} = (\dots, \overline{x}_1, \overline{x}_0)$ to $\lim_{n\to\infty} x_n^{p^n}$ for any lifts $x_n \in \mathbb{Z}_p[\mu_{p^\infty}]$ of $\overline{x}_n \in \mathbb{Z}_p[\mu_{p^\infty}]/(p)$. This map induces a surjective homomorphism $\theta : W^1_p(\mathbf{E}_{\mathbb{Q}_p}) \to \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$. Also,

$$\ker(\theta) = (z), \qquad z = \sum_{i=0}^{p-1} [1 + \overline{\pi}]^{i/p}.$$

To go backwards, show directly that base extension along $\theta$ induces an equivalence $\mathbf{F\acute{E}t}(W^1_p(\mathbf{E}_{\mathbb{Q}_p})) \cong \mathbf{F\acute{E}t}(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})$. (Key step: lift a $\mathbb{Z}/p\mathbb{Z}$-extension of a finite extension of $\mathbb{Q}_p(\mu_{p^\infty})$.)

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# Next steps

Right now, we have an equivalence

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \underleftarrow{W}_p^\dagger(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

What remains to be done (viewing $\mathbb{Q}_p^\times = p^{\mathbb{Z}} \cdot \mathbb{Z}_p^\times$):

- Adjoin non-$p$-power roots of unity, replacing $\mathbb{Z}_p^\times$ with $\widehat{\mathbb{Z}}^\times$.
- Pass from $p$-typical to big Witt vectors, replacing $p^{\mathbb{Z}}$ with $\mathbb{Q}^+$.
- Replace $\mathbf{E}_{\mathbb{Q}_p}$ with a characteristic 0 ring.

(Warning: $\mathbb{N}^+$ here denotes $\{1, 2, \dots\}$ viewed as a multiplicative monoid.)

# Adding roots of unity (relative $p$-adic Hodge theory)

Define the $\mathbb{Q}_p$-Banach algebra $A$ as the *uniform* completion of $\mathbb{Q}_p \otimes_{\mathbb{Z}} \mathbb{Z}[\mu_\infty]$ for the trivial norm on $\mathbb{Z}[\mu_\infty]$: complete not for the product seminorm $|\cdot|_\otimes$ but for its spectral seminorm $|x|_{\mathrm{sp}} = \lim_{n\to\infty} |x_n|_\otimes^{1/n}$. Put

$$A^+ = \{x \in A : |x|_{\mathrm{sp}} \leq 1\}, \quad \mathbf{E}_A^+ = \varprojlim A^+/(p), \quad \mathbf{E}_A = \mathbf{E}_A^+ \otimes_{\mathbf{E}_{\mathbb{Q}_p}^+} \mathbf{E}_{\mathbb{Q}_p}.$$

One can generalize the field of norms equivalence (K-Liu, Scholze):

$$\mathbf{F\acute{E}t}(A) \cong \mathbf{F\acute{E}t}(\mathbf{E}_A) \cong \mathbf{F\acute{E}t}(W_p^\dagger(\mathbf{E}_A)).$$

We thus obtain

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \varprojlim W_p^\dagger(\mathbf{E}_{\mathbb{Q}_p})\text{-algebras} \\ \text{with continuous } \mathbb{Q}_p^\times\text{-action} \end{matrix} \right\}.$$

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# Big Witt vectors

Recall that the functor **Ring** $\to$ **Set** given by $\bullet \rightsquigarrow \bullet^{\mathbb{N}^+}$ promotes uniquely to a functor $\mathbb{W}$ : **Ring** $\to$ **Ring** for which the formula

$$x = (x_n)_{n=1}^{\infty} \mapsto w(x) = \left( w_n(x) = \sum_{d|n} d x_d^{n/d} \right)_{n=1}^{\infty}$$

defines a natural transformation $w : \mathbb{W}(\bullet) \to \bullet^{\mathbb{N}^+}$. The monoid $\mathbb{N}^+$ acts via maps $F_n : \mathbb{W}(\bullet) \to \mathbb{W}(\bullet)$ for which

$$w_m(F_n(x)) = w_{mn}(x) \qquad (m, n \in \mathbb{N}^+; x \in \mathbb{W}(\bullet)).$$

The $F_n$ are typically neither injective nor surjective.

Arithmetic in the $n$-th component only depends on the $d$-th components for $d|n$; projecting onto $p$-power components recovers $W_p$ and $F_p$.

# Inverse limits and big Witt vectors

Put

$$\underleftarrow{\mathbb{W}}(\bullet) = \left\{ x = (x_{1/s})_{s \in \mathbb{N}^+} \in \mathbb{W}(\bullet)^{\mathbb{N}^+} : F_d(x_{1/sd}) = x_{1/s} \ (s, d \in \mathbb{N}^+) \right\}.$$

One can define ghost maps $w_t : \underleftarrow{\mathbb{W}}(\bullet) \to \bullet$ for all $t \in \mathbb{Q}^+$ by

$$w_t(x) = w_r(x_{1/s}) \qquad (r, s \in \mathbb{N}^+; r/s = t).$$

The maps $F_d$ define an action of the multiplicative monoid $\mathbb{Q}^+$ on $\underleftarrow{\mathbb{W}}(\bullet)$. If $\bullet$ carries an action of $\widehat{\mathbb{Z}}^\times$, we obtain an action of the finite idèles

$$\mathbb{A}_f^\times = \mathbb{Q}^+ \cdot \widehat{\mathbb{Z}}^\times.$$

# Norms on big Witt vectors

Define $|\cdot|_{\mathbb{W}} : \mathbb{W}(\mathbf{E}_A) \to [0, +\infty], |\cdot|_{\underleftarrow{\mathbb{W}}} : \underleftarrow{\mathbb{W}}(\mathbf{E}_A) \to [0, +\infty]$ by

$$|x|_{\mathbb{W}} = \sup_n \{|x_n|^{1/n}\}, \qquad |(x_{1/s})|_{\underleftarrow{\mathbb{W}}} = \sup_s \{|x_{1/s}|_{\mathbb{W}}^s\}.$$

Let $\mathbb{W}^0(\mathbf{E}_A), \underleftarrow{\mathbb{W}}^0(\mathbf{E}_A)$ be the subsets on which $|\cdot|_{\mathbb{W}}, |\cdot|_{\underleftarrow{\mathbb{W}}}$ are finite; these functions are then complete uniform norms. For $r > 0$, complete $\mathbb{W}^0(\mathbf{E}_A) \otimes_{\mathbb{Z}} \mathbb{Z}, \underleftarrow{\mathbb{W}}^0(\mathbf{E}_A) \otimes_{\mathbb{Z}} \mathbb{Z}$ for the product seminorms $|\cdot|_{\mathbb{W}}^r \otimes |\cdot|_p, |\cdot|_{\underleftarrow{\mathbb{W}}}^r \otimes |\cdot|_p$ to obtain $\mathbb{W}^r(\mathbf{E}_A), \underleftarrow{\mathbb{W}}^r(\mathbf{E}_A)$.

Put $\underleftarrow{\mathbb{W}}^\dagger(\mathbf{E}_A) = \cup_{r>0} \underleftarrow{\mathbb{W}}^r(\mathbf{E}_A)$. Then

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite} \quad \text{étale} \quad \underleftarrow{W}_p^\dagger(\mathbf{E}_A)\text{-algebras} \\ \text{with continuous } p^{\mathbb{Z}} \cdot \widehat{\mathbb{Z}}^\times\text{-action} \end{matrix} \right\}.$$

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$$|x|_{\mathbb{W}} = \sup_n \{|x_n|^{1/n}\}, \qquad |(x_{1/s})|_{\underleftarrow{\mathbb{W}}} = \sup_s \{|x_{1/s}|^s_{\mathbb{W}}\}.$$

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# Getting away from characteristic $p$

Define $|\cdot|_{\mathbb{W}} : \mathbb{W}(A) \to [0, +\infty], |\cdot|_{\underleftarrow{\mathbb{W}}} : \underleftarrow{\mathbb{W}}(A) \to [0, +\infty]$ by

$$|x|_{\mathbb{W}} = \sup_n \{|x_n|^{1/n}\}, \qquad |(x_{1/n})|_{\underleftarrow{\mathbb{W}}} = \sup_n \{|x_{1/n}|^n_{\mathbb{W}}\}.$$

Then $\underleftarrow{\mathbb{W}}^0(A) \to \underleftarrow{\mathbb{W}}^0(\mathbf{E}_A)$ is an isometric bijection for $|\cdot|_{\underleftarrow{\mathbb{W}}}$; this is analogous to the usual identification

$$\varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/(p).$$

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# Getting away from characteristic $p$

Define $|\cdot|_{\mathbb{W}} : \mathbb{W}(A) \to [0,+\infty], |\cdot|_{\underleftarrow{\mathbb{W}}} : \underleftarrow{\mathbb{W}}(A) \to [0,+\infty]$ by

$$|x|_{\mathbb{W}} = \sup_n \{|x_n|^{1/n}\}, \qquad |(x_{1/n})|_{\underleftarrow{\mathbb{W}}} = \sup_n \{|x_{1/n}|^n_{\mathbb{W}}\}.$$

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# Reducing the group action

Put $\mathbb{A}_f^\natural = \mathbb{N}^+ \cdot \widehat{\mathbb{Z}}^\times = \widehat{\mathbb{Z}} \cap \mathbb{A}_f^\times$. For $M$ a module over a ring on which $\mathbb{A}_f^\natural$ acts, an action of $\mathbb{A}_f^\natural$ on $M$ is *étale* if the action of every $d \in \mathbb{A}_f^\natural$ induces an isomorphism $d^* M \cong M$. (It is enough to check for $d \in \mathbb{N}^+$.)

We may rewrite the previous equivalence as

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For $p^{-n} \leq r$, the ghost map $w_{p^{-n}}$ extends to a map $\underleftarrow{\mathbb{W}}^r(A) \to A$. Hence the functor from right to left may be viewed as base change along $w_{p^{-n}}$ for any suitably large $n$ followed by Galois descent from $A$ to $\mathbb{Q}_p$. (The action of $\mathbb{N}^+$ ensures that the result does not depend on $n$.)

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# Eliminating the inverse limit

For $r > 0$, complete $\mathbb{W}^0(A) \otimes_{\mathbb{Z}} \mathbb{Z}$ for the product seminorm $|\cdot|_{\mathbb{W}}^r \otimes |\cdot|_p$ to obtain $\mathbb{W}^r(A)$. Put $\mathbb{W}^\dagger(A) = \cup_{r>0} \mathbb{W}^r(A)$. One then has

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Can one run backwards again by establishing this equivalence directly?

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# Not quite the Bost-Connes algebra

The integral Bost-Connes algebra $BC$ is the noncommutative ring generated by $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ plus elements $\mu_n, \mu_n^*$ for $n \in \mathbb{N}^+$ such that

$$\mu_{mn} = \mu_m \mu_n,\ \mu_{mn}^* = \mu_m^* \mu_n^* \qquad (m, n \in \mathbb{N}^+),$$
$$\mu_n^* \mu_n = 1 \qquad (n \in \mathbb{N}^+),$$
$$\mu_n \{r\} \mu_n^* = \frac{1}{n} \sum_{ns=r} \{s\} \qquad (n \in \mathbb{N}^+, r \in \mathbb{Q}/\mathbb{Z}).$$

After extending scalars from $\mathbb{Z}$ to $\mathbb{C}$, the time evolution by $t \in \mathbb{R}$ fixes $\mathbb{Z}[\mathbb{Q}/\mathbb{Z}]$ and takes $\mu_n$ to $n^{it}\mu_n$ for $n \in \mathbb{N}^+$.

This algebra admits symmetries by $\widehat{\mathbb{Z}}^\times$ acting on $\mathbb{Q}/\mathbb{Z}$. The enlarged algebra $BC^+ = BC[\widehat{\mathbb{Z}}^\times]$ acts on finite étale $\mathbb{W}^\dagger(A)$-algebras with continuous étale $\mathbb{A}_f^\natural$-action.

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# Globalization

Equip $\mathbb{Q}$ with the supremum of all of its absolute values (including the archimedean one!). Can one get an equivalence

$$\mathbf{F\acute{E}t}(\mathbb{Q}_p) \cong \left\{ \begin{matrix} \text{finite étale } \mathbb{W}^\dagger(\mathbb{Q}_p \widehat{\otimes}_{\mathbb{Z}} \mathbb{Z}[\mu_\infty])\text{-algebras} \\ \text{with continuous étale } \mathbb{A}_f^\natural\text{-action} \end{matrix} \right\}$$

by piecing together the local equivalences? (This question doesn't make sense for $\underleftarrow{\mathbb{W}}^\dagger$, because $|\cdot|_{\underleftarrow{\mathbb{W}}}$ doesn't behave well for archimedean norms: for $|\cdot|$ an archimedean norm, $|\cdot|^r$ is again an archimedean norm only for $r \in (0,1]$.)

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# $(\varphi, \Gamma)$-modules

Recall that

$$\left\{ \begin{matrix} \text{continuous representations of} \\ G_{\mathbb{Q}_p} \text{ on finite free } \mathbb{Z}_p\text{-modules} \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{étale } (\varphi, \Gamma)\text{-modules} \\ \text{over } \underleftarrow{W}^\dagger_p(\mathbf{E}_{\mathbb{Q}_p}) \end{matrix} \right\}.$$

Can one imitate the previous discussion so as to turn $(\varphi, \Gamma)$-modules into left modules over the $p$-adic component of some completion of $BC^+$ (as proposed on the previous slide)?

Partial answer: one can base-extend étale $(\varphi, \Gamma)$-modules from $\underleftarrow{W}^\dagger_p(\mathbf{E}_{\mathbb{Q}_p}) \cong \underleftarrow{W}^\dagger_p(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})$ to $W^\dagger_p(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})$ by projection. In fact, one gets a coherent family of modules over $W^r_p(\widehat{\mathbb{Q}_p(\mu_{p^\infty})})$ for all $r > 0$. But is this enough to get back to Galois representations?

# Motives and coefficients

Better yet, can one turn motives over $\mathbb{Q}$ into left modules over an appropriate completion of $BC^+$ in such a way that the $p$-adic component is the same as what one gets by changing base to $\mathbb{Q}_p$, taking $p$-adic étale cohomology, then proceeding as on the previous slide?

Such an object would seem to be an enhanced version of de Rham cohomology. Perhaps one can find it using Scholze's approach to the relative étale-to-de Rham comparison isomorphism?

And finally, does any of this lead to a spectral interpretation of $L$-functions?