Drinfeld's lemma for F-isocrystals

Kiran S. Kedlaya
work in progress with Daxin Xu (Morningside Center for Mathematics)

Department of Mathematics, University of California San Diego
kedlaya@ucsd.edu
These slides can be downloaded from https://kskedlaya.org/slides/.

Mathematical Congress of the Americas (virtual)
special session on Galois Representations and Automorphic Forms
July 13, 2021

Supported by NSF (grant DMS-1802161) and UC San Diego (Warschawski Professorship).

The UC San Diego campus sits on unceded ancestral land of the Kumeyaay Nation, who remain active and vital members of the San Diego community.

Contents

1. What is Drinfeld's lemma?
2. Convergent F-isocrystals
3. Drinfeld's lemma for convergent F-isocrystals
4. Overconvergent F-isocrystals and Drinfeld's lemma
5. Complements

Drinfeld's lemma

Throughout this talk, let p be a fixed prime; let X be a smooth $\mathbb{F}_p$-scheme; let k be an algebraically closed field of characteristic p; and let φ_k denote both the Frobenius map on k and its pullback to X_k (the base extension of X from $\mathbb{F}_p$ to k).

The metaprinciple underlying "Drinfeld's lemma" is that X shares many properties with the formal quotient of X_k by the action of the group generated by φ_k. That is, the natural morphism

$$X \to X_k/\varphi_k,$$

while not an isomorphism, does preserve many important structures.

Drinfeld's lemma

Throughout this talk, let p be a fixed prime; let X be a smooth $\mathbb{F}_p$-scheme; let k be an algebraically closed field of characteristic p; and let φ_k denote both the Frobenius map on k and its pullback to X_k (the base extension of X from $\mathbb{F}_p$ to k).

The metaprinciple underlying "Drinfeld's lemma" is that X shares many properties with the formal quotient of X_k by the action of the group generated by φ_k. That is, the natural morphism

$$X \to X_k/\varphi_k,$$

while not an isomorphism, does preserve many important structures.

Étale coverings

A concrete form of Drinfeld's lemma is that

$$\mathbf{FÉt}(X) \cong \mathbf{FÉt}(X_k/\varphi_k);$$

that is, the pullback functor from finite étale coverings of X to finite étale coverings of X_k equipped with isomorphisms with their φ_k-pullbacks is an equivalence of categories.

Another concrete form is that the category of quasicompact open immersions into X is equivalent (via base extension) to the category of quasicompact open immersions into X_k equipped with isomorphisms with their φ_k-pullbacks.

Étale coverings

A concrete form of Drinfeld's lemma is that

$$\mathbf{F\acute{E}t}(X) \cong \mathbf{F\acute{E}t}(X_k/\varphi_k);$$

that is, the pullback functor from finite étale coverings of X to finite étale coverings of X_k equipped with isomorphisms with their φ_k-pullbacks is an equivalence of categories.

Another concrete form is that the category of quasicompact open immersions into X is equivalent (via base extension) to the category of quasicompact open immersions into X_k equipped with isomorphisms with their φ_k-pullbacks.

Lisse and constructible sheaves

Let $\ell \neq p$ be a prime. From the previous statement, it follows that the categories of lisse $\mathbb{Q}_\ell$-sheaves on X and X_k/φ_k are equivalent, and similarly for the constructible derived categories (and with $\mathbb{Q}_\ell$ replaced by $\overline{\mathbb{Q}}_\ell$).

This is crucial to the construction of **excursion operators** in V. Lafforgue's approach to the Langlands correspondence for reductive groups over a global function field (see §8 of "Chtoucas pour les groupes réductifs…").

One can hope to simulate this approach using p-adic coefficients introduced by T. Abe in his transcription of L. Lafforgue's work for GL_n, but one needs an analogue of this form of Drinfeld's lemma. That is the goal of the present work.

Lisse and constructible sheaves

Let $\ell \neq p$ be a prime. From the previous statement, it follows that the categories of lisse $\mathbb{Q}_\ell$-sheaves on X and X_k/φ_k are equivalent, and similarly for the constructible derived categories (and with $\mathbb{Q}_\ell$ replaced by $\overline{\mathbb{Q}}_\ell$).

This is crucial to the construction of **excursion operators** in V. Lafforgue's approach to the Langlands correspondence for reductive groups over a global function field (see §8 of "Chtoucas pour les groupes réductifs...").

One can hope to simulate this approach using p-adic coefficients introduced by T. Abe in his transcription of L. Lafforgue's work for GL_n, but one needs an analogue of this form of Drinfeld's lemma. That is the goal of the present work.

Lisse and constructible sheaves

Let $\ell \neq p$ be a prime. From the previous statement, it follows that the categories of lisse $\mathbb{Q}_\ell$-sheaves on X and X_k/φ_k are equivalent, and similarly for the constructible derived categories (and with $\mathbb{Q}_\ell$ replaced by $\overline{\mathbb{Q}}_\ell$).

This is crucial to the construction of **excursion operators** in V. Lafforgue's approach to the Langlands correspondence for reductive groups over a global function field (see §8 of "Chtoucas pour les groupes réductifs...").

One can hope to simulate this approach using p-adic coefficients introduced by T. Abe in his transcription of L. Lafforgue's work for GL_n, but one needs an analogue of this form of Drinfeld's lemma. That is the goal of the present work.

Contents

1 What is Drinfeld's lemma?

2 Convergent F-isocrystals

3 Drinfeld's lemma for convergent F-isocrystals

4 Overconvergent F-isocrystals and Drinfeld's lemma

5 Complements

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category $\mathbf{F\text{-}Isoc}(X)$ of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category $\mathbf{F\text{-}Isoc}(X)$ of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent** F**-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category $\mathbf{F}\text{-}\mathbf{Isoc}(X)$ of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category **F-Isoc**(X) of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category **F-Isoc**(X) of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category **F-Isoc**(X) of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Convergent F-isocrystals via formal lifts

For simplicity, suppose now that X is affine. Let P be a smooth formal scheme over $\mathbb{Z}_p$ with special fiber X.

A **convergent F-isocrystal** on X consists of:

- a vector bundle $\mathcal{E}$ on the Raynaud generic fiber $P_{\mathbb{Q}_p}$;
- an integrable connection ∇ on $\mathcal{E}$;
- an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ of vector bundles with connection, where $\sigma : P \to P$ is some lift of absolute Frobenius on X.

It can be shown that the category **F-Isoc**(X) of convergent F-isocrystals is naturally independent of the choices of P and σ; thus the definition extends to nonaffine X.

We may similarly define convergent F-isocrystals on a smooth scheme over any perfect field of characteristic p, and in particular on X_k.

Newton polygons and slope filtrations

Objects of **F-Isoc**(Spec k) are uniquely classified by their **Newton polygons** by Dieudonné–Manin.

Theorem (Grothendieck–Katz, 1970s)

*For $\mathcal{E} \in$ **F-Isoc**(X), the Newton polygon defines an upper semicontinuous function on $|X|$. Moreover, if this function is constant, then $\mathcal{E}$ admits a unique filtration*

$$0 = \mathcal{E}_0 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

in which each successive quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ has the unique slope μ_i and $\mu_1 < \cdots < \mu_l$.

Unit-root objects and Galois representations

An object $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$ is **unit-root** (or **étale**) if its Newton polygon is constant with all slopes 0.

Theorem (Katz–Crew, 1980s)

There is an equivalence of tensor categories between the unit-root objects of $\mathbf{F\text{-}Isoc}(X)$ and the category of continuous representations of $\pi_1(X, \overline{x})$ (for a fixed geometric point $\overline{x}$) on finite-dimensional $\mathbb{Q}_p$-vector spaces.

One has a similar assertion for objects whose Newton polygons is constant with all slopes equal to some fixed nonzero value.

Unit-root objects and Galois representations

An object $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$ is **unit-root** (or **étale**) if its Newton polygon is constant with all slopes 0.

Theorem (Katz–Crew, 1980s)

There is an equivalence of tensor categories between the unit-root objects of $\mathbf{F\text{-}Isoc}(X)$ *and the category of continuous representations of* $\pi_1(X, \overline{x})$ *(for a fixed geometric point* $\overline{x}$*) on finite-dimensional* $\mathbb{Q}_p$*-vector spaces.*

One has a similar assertion for objects whose Newton polygons is constant with all slopes equal to some fixed nonzero value.

Full faithfulness of restriction

Theorem (K, 2000s)

Let U be an open dense subset of X. Then the restriction functor $\mathbf{F}\text{-}\mathbf{Isoc}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(U)$ *is fully faithful.*

In the analogy with lisse sheaves, this corresponds to the fact that $\pi_1(X, \overline{x})$ is a quotient of $\pi_1(U, \overline{x})$. However, the p-adic statement cannot be deduced from this; it requires a dedicated argument drawn from de Jong's work on crystalline Dieudonné theory.

Full faithfulness of restriction

Theorem (K, 2000s)

Let U be an open dense subset of X. Then the restriction functor $\mathbf{F}\text{-}\mathbf{Isoc}(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(U)$ *is fully faithful.*

In the analogy with lisse sheaves, this corresponds to the fact that $\pi_1(X, \overline{x})$ is a quotient of $\pi_1(U, \overline{x})$. However, the p-adic statement cannot be deduced from this; it requires a dedicated argument drawn from de Jong's work on crystalline Dieudonné theory.

Contents

1. What is Drinfeld's lemma?
2. Convergent F-isocrystals
3. Drinfeld's lemma for convergent F-isocrystals
4. Overconvergent F-isocrystals and Drinfeld's lemma
5. Complements

F-isocrystals on a Frobenius quotient

Let $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ be the category consisting of convergent F-isocrystals on X_k equipped with isomorphisms with their φ_k-pullbacks.

There is a natural pullback functor $\mathbf{F\text{-}Isoc}(X) \to \mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ which is **not** an equivalence of categories. For example, its essential image does not contain any external product

$$\mathcal{E} \boxtimes \mathcal{F}$$

in which $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$, $\mathcal{F} \in \mathbf{F\text{-}Isoc}(\operatorname{Spec} k)$, and $\mathcal{F}$ is not unit-root.

What we'd like to show is that in some sense, this is the worst that can happen.

F-isocrystals on a Frobenius quotient

Let $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ be the category consisting of convergent F-isocrystals on X_k equipped with isomorphisms with their φ_k-pullbacks.

There is a natural pullback functor $\mathbf{F\text{-}Isoc}(X) \to \mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ which is **not** an equivalence of categories. For example, its essential image does not contain any external product

$$\mathcal{E} \boxtimes \mathcal{F}$$

in which $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$, $\mathcal{F} \in \mathbf{F\text{-}Isoc}(\operatorname{Spec} k)$, and $\mathcal{F}$ is not unit-root.

What we'd like to show is that in some sense, this is the worst that can happen.

F-isocrystals on a Frobenius quotient

Let $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ be the category consisting of convergent F-isocrystals on X_k equipped with isomorphisms with their φ_k-pullbacks.

There is a natural pullback functor $\mathbf{F\text{-}Isoc}(X) \to \mathbf{F\text{-}Isoc}(X_k/\varphi_k)$ which is **not** an equivalence of categories. For example, its essential image does not contain any external product

$$\mathcal{E} \boxtimes \mathcal{F}$$

in which $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$, $\mathcal{F} \in \mathbf{F\text{-}Isoc}(\operatorname{Spec} k)$, and $\mathcal{F}$ is not unit-root.

What we'd like to show is that in some sense, this is the worst that can happen.

Stratification by the (diagonal) Newton polygon

Given an object $\mathcal{E}$ of $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$, the underlying object of $\mathbf{F\text{-}Isoc}(X_k)$ has a stratification by Newton polygons, which must be stable under the action of φ_k. Using an earlier version of Drinfeld's lemma, we see that this stratification is itself pulled back from X.

Suppose further that the Newton polygon is constant. Then the slope flitration of $\mathcal{E}$ is also stable under the action of φ_k.

Suppose further that $\mathcal{E}$ is unit-root. Then $\mathcal{E}$ corresponds to a p-adic representation of $\pi_1(X, \overline{x})$ with an additional action of φ_k.

Stratification by the (diagonal) Newton polygon

Given an object $\mathcal{E}$ of $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$, the underlying object of $\mathbf{F\text{-}Isoc}(X_k)$ has a stratification by Newton polygons, which must be stable under the action of φ_k. Using an earlier version of Drinfeld's lemma, we see that this stratification is itself pulled back from X.

Suppose further that the Newton polygon is constant. Then the slope flitration of $\mathcal{E}$ is also stable under the action of φ_k.

Suppose further that $\mathcal{E}$ is unit-root. Then $\mathcal{E}$ corresponds to a p-adic representation of $\pi_1(X, \overline{x})$ with an additional action of φ_k.

Stratification by the (diagonal) Newton polygon

Given an object $\mathcal{E}$ of $\mathbf{F\text{-}Isoc}(X_k/\varphi_k)$, the underlying object of $\mathbf{F\text{-}Isoc}(X_k)$ has a stratification by Newton polygons, which must be stable under the action of φ_k. Using an earlier version of Drinfeld's lemma, we see that this stratification is itself pulled back from X.

Suppose further that the Newton polygon is constant. Then the slope flitration of $\mathcal{E}$ is also stable under the action of φ_k.

Suppose further that $\mathcal{E}$ is unit-root. Then $\mathcal{E}$ corresponds to a p-adic representation of $\pi_1(X, \overline{x})$ with an additional action of φ_k.

Relative Dieudonné–Manin

Theorem

*Every object $\mathcal{E} \in$ **F-Isoc**(X_k/φ_k) decomposes uniquely as a direct sum*

$$\mathcal{E} \cong \bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$$

*in which for $d = \frac{r}{s}$ in lowest terms, $\mathcal{E}_d$ is obtained by pulling back an object of **F-Isoc**(X) equipped with an endomorphism F such that $F^s = p^r$, which then gives the action of φ_k on the pullback. (The latter may be recovered from $\mathcal{E}_d$ as the kernel of $\varphi_k^s - p^r$.)*

Contents

1. What is Drinfeld's lemma?
2. Convergent F-isocrystals
3. Drinfeld's lemma for convergent F-isocrystals
4. Overconvergent F-isocrystals and Drinfeld's lemma
5. Complements

Why not convergent F-isocrystals?

Convergent F-isocrystals do not have good cohomological properties: for $\mathcal{E} \in \mathbf{F\text{-}Isoc}(X)$, the spaces $H^i(X, \mathcal{E})$ (defined using de Rham cohomology) are infinite-dimensional over $\mathbb{Q}_p$.

The essential issue is that

$$\frac{d}{dT} : \mathbb{Q}_p\langle T\rangle \to \mathbb{Q}_p\langle T\rangle$$

is not surjective: antidifferentiation does not preserve convergence at the boundary of a closed disc.

Construction of overconvergent F-isocrystals

We may define the category $\mathbf{F\text{-}Isoc}^\dagger(X)$ of **overconvergent F-isocrystals** by taking the definition of convergent F-isocrysals and replacing the formal lift P with a **weak formal lift** $P^\dagger$, choosing the Frobenius lift σ so that it acts on $P^\dagger$.

For example, if $X = \mathbb{A}^n_{\mathbb{F}_p}$ and $\Gamma(P, \mathcal{O}) = \mathbb{Z}_p\langle x_1, \dots, x_n \rangle$ (the ring of power series convergent on the closed unit polydisc), then $\Gamma(P^\dagger, \mathcal{O}) = \mathbb{Z}_p\langle x_1, \dots, x_n \rangle^\dagger$ (the ring of power series convergent on a polydisc of **some** radius strictly greater than 1, which may depend on the series). In general, $\Gamma(P^\dagger, \mathcal{O})$ will be a quotient of a ring of this form and $\Gamma(P, \mathcal{O})$ will be its p-adic completion.

Construction of overconvergent F-isocrystals

We may define the category $\mathbf{F\text{-}Isoc}^\dagger(X)$ of **overconvergent F-isocrystals** by taking the definition of convergent F-isocrysals and replacing the formal lift P with a **weak formal lift** $P^\dagger$, choosing the Frobenius lift σ so that it acts on $P^\dagger$.

For example, if $X = \mathbb{A}^n_{\mathbb{F}_p}$ and $\Gamma(P, \mathcal{O}) = \mathbb{Z}_p\langle x_1, \dots, x_n\rangle$ (the ring of power series convergent on the closed unit polydisc), then $\Gamma(P^\dagger, \mathcal{O}) = \mathbb{Z}_p\langle x_1, \dots, x_n\rangle^\dagger$ (the ring of power series convergent on a polydisc of **some** radius strictly greater than 1, which may depend on the series). In general, $\Gamma(P^\dagger, \mathcal{O})$ will be a quotient of a ring of this form and $\Gamma(P, \mathcal{O})$ will be its p-adic completion.

Convergent vs. overconvergent

Theorem (K, 2000s)

The restriction functor $\mathbf{F\text{-}Isoc}^\dagger(X) \to \mathbf{F\text{-}Isoc}(X)$ *is fully faithful.*

Warning: It is not true that an irreducible object in $\mathbf{F\text{-}Isoc}^\dagger(X)$ remains irreducible in $\mathbf{F\text{-}Isoc}(X)$! For example, an object in $\mathbf{F\text{-}Isoc}^\dagger(X)$ with constant Newton polygon does not in general admit a slope filtration in $\mathbf{F\text{-}Isoc}^\dagger(X)$, but it does have one in $\mathbf{F\text{-}Isoc}(X)$.

This state of affairs can be described nicely in terms of Tannakian monodormy groups (Crew, D'Addezio).

Convergent vs. overconvergent

Theorem (K, 2000s)

The restriction functor $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$ *is fully faithful.*

Warning: It is not true that an irreducible object in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ remains irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$! For example, an object in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ with constant Newton polygon does not in general admit a slope filtration in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$, but it does have one in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$.

This state of affairs can be described nicely in terms of Tannakian monodormy groups (Crew, D'Addezio).

Convergent vs. overconvergent

Theorem (K, 2000s)

The restriction functor $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \to \mathbf{F}\text{-}\mathbf{Isoc}(X)$ *is fully faithful.*

Warning: It is not true that an irreducible object in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ remains irreducible in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$! For example, an object in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ with constant Newton polygon does not in general admit a slope filtration in $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$, but it does have one in $\mathbf{F}\text{-}\mathbf{Isoc}(X)$.

This state of affairs can be described nicely in terms of Tannakian monodormy groups (Crew, D'Addezio).

Relative Dieudonné–Manin

Theorem

Every object $\mathcal{E} \in \mathbf{F\text{-}Isoc}^{\dagger}(X_k/\varphi_k)$ decomposes uniquely as a direct sum

$$\mathcal{E} \cong \bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$$

in which for $d = \frac{r}{s}$ in lowest terms, $\mathcal{E}_d$ is obtained by pulling back an object of $\mathbf{F\text{-}Isoc}^{\dagger}(X)$ equipped with an endomorphism F such that $F^s = p^r$, which then gives the action of φ_k on the pullback. (The latter may be recovered from $\mathcal{E}_d$ as the kernel of $\varphi_k^s - p^r$.)

Using full faithfulness of restriction, this reduces to the convergent case.

Relative Dieudonné–Manin

Theorem

Every object $\mathcal{E} \in \mathbf{F\text{-}Isoc}^{\dagger}(X_k/\varphi_k)$ decomposes uniquely as a direct sum

$$\mathcal{E} \cong \bigoplus_{d \in \mathbb{Q}} \mathcal{E}_d$$

in which for $d = \frac{r}{s}$ in lowest terms, $\mathcal{E}_d$ is obtained by pulling back an object of $\mathbf{F\text{-}Isoc}^{\dagger}(X)$ equipped with an endomorphism F such that $F^s = p^r$, which then gives the action of φ_k on the pullback. (The latter may be recovered from $\mathcal{E}_d$ as the kernel of $\varphi_k^s - p^r$.)

Using full faithfulness of restriction, this reduces to the convergent case.

Contents

1 What is Drinfeld's lemma?

2 Convergent F-isocrystals

3 Drinfeld's lemma for convergent F-isocrystals

4 Overconvergent F-isocrystals and Drinfeld's lemma

5 Complements

Arithmetic $\mathcal{D}$-modules

Objects of $\mathbf{F\text{-}Isoc}^\dagger(X)$ can be viewed as left modules for a certain topological ring $\mathcal{D}^\dagger_{P,\mathbb{Q}}$ of differential operators on the weak formal lift P of X. These* are the p-adic analogue of lisse $\overline{\mathbb{Q}}_\ell$-sheaves; the analogue of the constructible derived category has been described by Abe using Berthelot's **arithmetic $\mathcal{D}$-modules**.

As in the ℓ-adic setting, one can formally promote Drinfeld's lemma for overconvergent F-isocrystals to a corresponding statement about the derived category of holonomic arithmetic $\mathcal{D}$-modules.

The key point is that given an object of this category over X_k, one has a canonical stratification on each stratum of which we get a lisse object, and that stratification (being stable under φ_k) must descend to X by the "usual" Drinfeld's lemma.

*More precisely, objects of the formal base extension $\mathbf{F\text{-}Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

Arithmetic $\mathcal{D}$-modules

Objects of $\mathbf{F\text{-}Isoc}^\dagger(X)$ can be viewed as left modules for a certain topological ring $\mathcal{D}^\dagger_{P,\mathbb{Q}}$ of differential operators on the weak formal lift P of X. These* are the p-adic analogue of lisse $\overline{\mathbb{Q}}_\ell$-sheaves; the analogue of the constructible derived category has been described by Abe using Berthelot's **arithmetic $\mathcal{D}$-modules**.

As in the ℓ-adic setting, one can formally promote Drinfeld's lemma for overconvergent F-isocrystals to a corresponding statement about the derived category of holonomic arithmetic $\mathcal{D}$-modules.

The key point is that given an object of this category over X_k, one has a canonical stratification on each stratum of which we get a lisse object, and that stratification (being stable under φ_k) must descend to X by the "usual" Drinfeld's lemma.

*More precisely, objects of the formal base extension $\mathbf{F\text{-}Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

Arithmetic $\mathcal{D}$-modules

Objects of $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X)$ can be viewed as left modules for a certain topological ring $\mathcal{D}^\dagger_{P,\mathbb{Q}}$ of differential operators on the weak formal lift P of X. These* are the p-adic analogue of lisse $\overline{\mathbb{Q}}_\ell$-sheaves; the analogue of the constructible derived category has been described by Abe using Berthelot's **arithmetic $\mathcal{D}$-modules**.

As in the ℓ-adic setting, one can formally promote Drinfeld's lemma for overconvergent F-isocrystals to a corresponding statement about the derived category of holonomic arithmetic $\mathcal{D}$-modules.

The key point is that given an object of this category over X_k, one has a canonical stratification on each stratum of which we get a lisse object, and that stratification (being stable under φ_k) must descend to X by the "usual" Drinfeld's lemma.

*More precisely, objects of the formal base extension $\mathbf{F}\text{-}\mathbf{Isoc}^\dagger(X) \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p$.

Larger products

For $i = 1, \ldots, n$, let X_i be a smooth scheme over a perfect field k_i. One can then state a form of Drinfeld's lemma for convergent and overconvergent F-isocrystals on the scheme

$$X = X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$$

equipped with actions of all but one of the partial Frobenius maps. (Or more symmetrically, one can use all of them and require that the composition agrees with the action of total Frobenius.)

Roughly speaking, this asserts that

$$\pi_1(X/\langle \varphi_1, \ldots, \varphi_{n-1} \rangle) \cong \pi_1(X_1) \times \cdots \times \pi_1(X_n);$$

this can be made precise in terms of Tannakian fundamental groups.

Larger products

For $i = 1, \ldots, n$, let X_i be a smooth scheme over a perfect field k_i. One can then state a form of Drinfeld's lemma for convergent and overconvergent F-isocrystals on the scheme

$$X = X_1 \times_{\mathbb{F}_p} \cdots \times_{\mathbb{F}_p} X_n$$

equipped with actions of all but one of the partial Frobenius maps. (Or more symmetrically, one can use all of them and require that the composition agrees with the action of total Frobenius.)

Roughly speaking, this asserts that

$$\pi_1(X/\langle \varphi_1, \ldots, \varphi_{n-1} \rangle) \cong \pi_1(X_1) \times \cdots \times \pi_1(X_n);$$

this can be made precise in terms of Tannakian fundamental groups.

Thank you for your attention!

¡Gracias por su atención!

Merci pour votre attention!

Obrigado pela sua atenção!