

Roots of unity in geometry (and elsewhere)

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*The UC San Diego campus occupies unceded ancestral homelands of the Kumeyaay Nation. The Kumeyaay people continue to have an important and thriving presence in the region.

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Statement of IMO 1990-6

Problem

Prove that there exists a convex 1990-gon with the following two properties:

- (a) *All angles are equal.*
- (b) *The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \dots, 1990^2$ in some order.*

The key to solving this is to realize that this is **not** a geometry problem!

Additive relations among roots of unity

Let N be a positive integer and set $\zeta_N := e^{2\pi i/N} \in \mathbb{C}$. For each proper divisor d of N and each $i \in \{0, \dots, d-1\}$, we have

$$\zeta_N^i + \zeta_N^{i+d} + \zeta_N^{i+2d} + \dots + \zeta_N^{i+(N/d-1)d} = 0.$$

Important note: this is false for $d = N$!

Consequently, if $a_0, \dots, a_{N-1} \in \mathbb{Q}$ is a d -**periodic** sequence (i.e., a_j depends only on $j \pmod{d}$) for some proper divisor d of N , then $a_0 + a_1\zeta_N + a_2\zeta_N^2 + \dots + a_{N-1}\zeta_N^{N-1} = 0$.

By the same token, if the tuple (a_0, \dots, a_{N-1}) is a \mathbb{Q} -linear combination of tuples, each of which is d -periodic for some proper divisor d of N , then $a_0 + a_1\zeta_N + a_2\zeta_N^2 + \dots + a_{N-1}\zeta_N^{N-1} = 0$.

Exercise: solve the problem from here! **Hint:** First try replacing 1990 with pq (p, q prime) and get side lengths $1, 2, \dots, pq$ in some order.

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Classification of additive relations among roots of unity

Theorem

For any $a_0, \dots, a_{N-1} \in \mathbb{Q}$,

$$a_0 + a_1\zeta_N + a_2\zeta_N^2 + \dots + a_{N-1}\zeta_N^{N-1} = 0$$

if and only if (a_0, \dots, a_{N-1}) is a \mathbb{Q} -linear combination of N -tuples, each of which is d -periodic for some proper divisor d of N .

Sketch of the proof:

- Prove that the dimension of the \mathbb{Q} -vector space generated by the known relations is at least $N - \varphi(N)$. **Hint:** the d -periodic sequences generate a space of dimension d ; now use inclusion-exclusion.
- Prove that there are no nonzero relations among $1, \zeta_N, \dots, \zeta_N^{\varphi(N)-1}$. See next slide.

Cyclotomic polynomials

Define the N -th **cyclotomic polynomial** as

$$\Phi_N(x) = \prod_{\substack{i=0 \\ \gcd(i,N)=1}}^{N-1} (x - \zeta_N^i) = \prod_{d|N} (x^d - 1)^{\mu(N/d)}.$$

The first formula gives a polynomial of degree $\varphi(N)$ over \mathbb{C} ; the second shows that the coefficients are in \mathbb{Z} (by comparing power series over \mathbb{Z}).

Since $\Phi_N(x)$ has ζ_N as a (simple) root, its coefficients give an additive relation among $1, \zeta_N, \dots, \zeta_N^{\varphi(N)}$ with coefficients in \mathbb{Z} .

In fact $\Phi_N(x)$ is **irreducible** over \mathbb{Q} , so there is no nonzero \mathbb{Q} -relation among $1, \zeta_N, \dots, \zeta_N^{\varphi(N)-1}$. This can be proved using **Gauss's lemma** plus formal properties of polynomials with mod- p coefficients; see next slide.

Irreducibility of cyclotomic polynomials (sketch)

There is a unique monic irreducible factor $f(x)$ of $\Phi_N(x)$ with $f(\zeta_N) = 0$.

Claim: for every prime p not dividing N , the roots of f are closed under $x \mapsto x^p$; this will imply $\deg(f) \geq \varphi(N)$ (why?) and so $f = \Phi_N$.

Put $g(x) = (x^N - 1)/f(x)$. If my claim fails, then $g(x^p)$ and $f(x)$ have a root in common, so in fact $g(x^p) = f(x)h(x)$ for some polynomial h with rational **integer** coefficients (Gauss's lemma).

Now reduce everything modulo p , using bars to denote the reduction:

$$\bar{f}(x)\bar{h}(x) = \bar{g}(x^p) = \bar{g}(x)^p. \quad (!!)$$

This means that \bar{f} and \bar{g} have a common factor, so $\bar{f}\bar{g}$ has a **repeated** factor. But this is impossible because for $'$ the formal derivative,

$$(\bar{f}\bar{g})'(x) = \overline{(fg)'(x)} = Nx^{N-1}$$

has no common factor with $\bar{f}\bar{g}(x) = \bar{f}\bar{g}(x) = x^N - 1$.

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Short additive relations among roots of unity: an example

Let ζ_1, \dots, ζ_6 be roots of unity that sum to zero. Then either:

- they cancel in pairs;
- they form two triples, each of the form $\zeta, e^{2\pi i/3}\zeta, e^{4\pi i/3}\zeta$ for some ζ ;
- or they have the form $-\zeta e^{2\pi i/3}, -\zeta e^{4\pi i/3}, \zeta e^{2\pi i/5}, \zeta e^{4\pi i/5}, \zeta e^{6\pi i/5}, \zeta e^{8\pi i/5}$ for some ζ .

Exercise: prove this using what we have already discussed! **Hint:** one approach uses the following lemma.

Lemma (Conway–Jones, 1974)

Let ζ_1, \dots, ζ_n be powers of $\zeta_N = e^{2\pi i/N}$ which sum to zero. Assume also:

- there is no nonempty proper subset S of $\{1, \dots, n\}$ such that $\sum_{i \in S} \zeta_i = 0$ (that is, we have an **indecomposable** additive relation);
- there is no root of unity ζ such that $\zeta\zeta_1, \dots, \zeta\zeta_n$ all have order dividing d for some proper divisor d of N .

Then $n \geq 2 + \sum_{p|N} (p - 2)$ where p runs over distinct prime factors of N .

Short additive relations among roots of unity: more results

Indecomposable additive relations among ζ_1, \dots, ζ_n have been classified:

- $n \leq 6$: Mann (1975)
- $n \leq 8$: Włodarski (1969)
- $n \leq 9$: Conway–Jones (1976)
- $n \leq 12$: Poonen[†]–Rubinstein (1998), and independently Lisovyy–Tykhyi (2014)
- $n \leq 21$: Christie–Dykema–Klep[‡] (2020 preprint)
- $n = 24$: Fu (2022)

The last two arguments were heavily computer-assisted. One can probably go a bit further, but the complexity seems to be exponential in n .

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Some applications of this classification

- Counting the intersections of diagonals of a regular polygon (Poonen–Rubinstein)
- The orchard problem: choose n points in the plane to maximize the number of lines through exactly 3 points (Green[§]–Tao[¶])
- Classification of Frobenius–Perron dimensions of fusion categories (Calegari^{||}–Morrison–Snyder)
- Algebraic solutions of the Painlevé VI differential equation (Lisovyy–Tykhyy)
- Computing the possible proportions of elements of a finite group which vanish on some irreducible character (Zeng–Yang–Dolfin)

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Tetrahedra with rational dihedral angles

We call an angle **rational** if its radian measure is a rational multiple of π . In a triangle, if two angles are rational then so is the third.

By contrast, in a tetrahedron, it is not so easy to force all six **dihedral** angles to be rational. (We'll see the algebraic relationship among these angles later.)

One reason why we care: any two convex polygons in the plane with the same area are **scissors-congruent**. However...

Theorem (Dehn–Sydler)

*A tetrahedron is scissors-congruent to a cube of the same volume if and only if the **Dehn invariant** is zero.*

For example, the Dehn invariant is nonzero for a regular tetrahedron (Hilbert's 3rd problem); but it is zero for a tetrahedron with rational dihedral angles.

Tetrahedra with rational dihedral angles: classification

In their 1976 paper, Conway–Jones asked for a classification of tetrahedra with rational dihedral angles. Here is the answer! (Besides [our preprint](#), see also [this writeup in Quanta](#).)

Theorem (Kedlaya–Kolpakov–Poonen–Rubinstein, 2020 preprint)

Up to symmetry, any tetrahedron in \mathbb{R}^3 with all dihedral angles rational is either one of 59 sporadic examples (next slide) or has one of the forms

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x \right) \quad \text{for } \frac{\pi}{6} < x < \frac{\pi}{2},$$

$$\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x \right) \quad \text{for } \frac{\pi}{6} < x \leq \frac{\pi}{3}.$$

Convention: we list dihedral angles in the order $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$, where α_{ij} means the angle between faces i and j .

Sporadic tetrahedra (key on the next slide)

N	$(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ as multiples of π/N
12	$(3, 4, 3, 4, 6, 8) = H_2(\pi/4)$
24	$(5, 9, 6, 8, 13, 15)$
12	$(3, 6, 4, 6, 4, 6) = T_0$
24	$(7, 11, 7, 13, 8, 12)$
15	$(3, 3, 3, 5, 10, 10) = T_{18}, (2, 4, 4, 4, 10, 10), (3, 3, 4, 4, 9, 11)$
15	$(3, 3, 5, 5, 9, 9) = T_7$
15	$(5, 5, 5, 9, 6, 6) = T_{23}, (3, 7, 6, 6, 7, 7), (4, 8, 5, 5, 7, 7)$
21	$(3, 9, 7, 7, 12, 12), (4, 10, 6, 6, 12, 12), (6, 6, 7, 7, 9, 15)$
30	$(6, 12, 10, 15, 10, 20) = T_{17}, (4, 14, 10, 15, 12, 18)$
60	$(8, 28, 19, 31, 25, 35), (12, 24, 15, 35, 25, 35), (13, 23, 15, 35, 24, 36), (13, 23, 19, 31, 20, 40)$
30	$(6, 18, 10, 10, 15, 15) = T_{13}, (4, 16, 12, 12, 15, 15), (9, 21, 10, 10, 12, 12)$
30	$(6, 6, 10, 12, 15, 20) = T_{16}, (5, 7, 11, 11, 15, 20)$
60	$(7, 17, 20, 24, 35, 35), (7, 17, 22, 22, 33, 37), (10, 14, 17, 27, 35, 35), (12, 12, 17, 27, 33, 37)$
30	$(6, 10, 10, 15, 12, 18) = T_{21}, (5, 11, 10, 15, 13, 17)$
60	$(10, 22, 21, 29, 25, 35), (11, 21, 19, 31, 26, 34), (11, 21, 21, 29, 24, 36), (12, 20, 19, 31, 25, 35)$
30	$(6, 10, 6, 10, 15, 24) = T_6$
60	$(7, 25, 12, 20, 35, 43)$
30	$(6, 12, 6, 12, 15, 20) = T_2$
60	$(12, 24, 13, 23, 29, 41)$
30	$(6, 12, 10, 10, 15, 18) = T_3, (7, 13, 9, 9, 15, 18)$
60	$(12, 24, 17, 23, 33, 33), (14, 26, 15, 21, 33, 33), (15, 21, 20, 20, 27, 39), (17, 23, 18, 18, 27, 39)$
30	$(6, 15, 6, 18, 10, 20) = T_4, (6, 15, 7, 17, 9, 21)$
60	$(9, 33, 14, 34, 21, 39), (9, 33, 15, 33, 20, 40), (11, 31, 12, 36, 21, 39), (11, 31, 15, 33, 18, 42)$
30	$(6, 15, 10, 15, 12, 15) = T_1, (6, 15, 11, 14, 11, 16), (8, 13, 8, 17, 12, 15),$ $(8, 13, 9, 18, 11, 14), (8, 17, 9, 12, 11, 16), (9, 12, 9, 18, 10, 15)$
30	$(10, 12, 10, 12, 15, 12) = T_5$
60	$(19, 25, 20, 24, 29, 25)$

How to read the table

Each tetrahedron is represented by an integer N and a list of six integers, representing $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$ as multiples of $\frac{\pi}{N}$.

The extra labels indicate examples of tetrahedra previously known to be rectifiable.

The groups between horizontal lines are orbits for a certain “extra” symmetry group (more on this shortly).

Regge symmetry

In the 1960s, two physicists studying angular momentum in quantum mechanics discovered an amazing fact about tetrahedra.

Theorem (Ponzano–Regge)

For any tetrahedron with edges $(\ell_{12}, \ell_{34}, \ell_{13}, \ell_{24}, \ell_{14}, \ell_{23})$ and dihedral angles $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$, there is another with edges

$$(\ell_{12}, \ell_{34}, s - \ell_{13}, s - \ell_{24}, s - \ell_{14}, s - \ell_{23}), \quad s = \frac{1}{2}(\ell_{13} + \ell_{24} + \ell_{14} + \ell_{23})$$

and dihedral angles

$$(\alpha_{12}, \alpha_{34}, \sigma - \alpha_{13}, \sigma - \alpha_{24}, \sigma - \alpha_{14}, \sigma - \alpha_{23}), \quad \sigma = \frac{1}{2}(\alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23}).$$

In 1999, Roberts observed that the Dehn invariant is also preserved. In 2019, Akopyan–Izmestiev gave a “classical” proof of the theorem.

Consequences of Regge symmetry

The family of tetrahedra with dihedral angles $(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x)$ was discovered by Hill in 1895. Applying a Regge symmetry gives the family $(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x)$.

Together with the action of S_4 on faces, the Regge symmetry generates a larger group acting on isomorphism classes of tetrahedra. Our table of sporadic tetrahedra indicates orbits for this larger group.

In particular, all of the sporadic examples are “explained” by the classical ones via this larger symmetry group **except** for the ones with $N = 21$.

Rational-angle line configurations

It is also natural to consider degenerate cases.

Problem

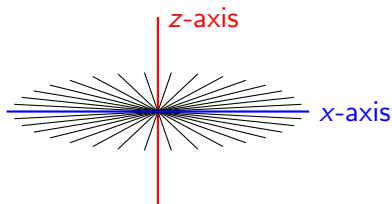
*Find all sets of lines through the origin in \mathbb{R}^3 , any two of which form a rational angle. We call such a set a **rational-angle line configuration**.*

Of course, we consider these sets up to isometries of \mathbb{R}^3 fixing the origin (rotation, reflection). Also, it is enough to classify **maximal** sets with this property (i.e., sets to which no additional line can be added).

For example, the three coordinate axes form angles of $\frac{\pi}{2}$, but this is not maximal...

A maximal configuration

Consider all of the lines in the xy -plane that form a rational angle with the x -axis, together with the z -axis. This is a rational-angle line configuration.

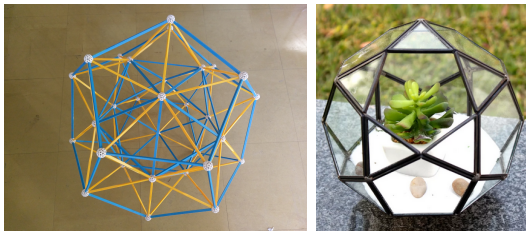


Exercise: show that this is in fact maximal.

We will see this is the only **infinite** maximal configuration.

Another maximal configuration

Consider a dodecahedron, and draw the 15 lines from the center to the midpoints of each of the 30 edges. (These are also the midpoints of the edges of an icosahedron, or the vertices of an **icosidodecahedron**.)



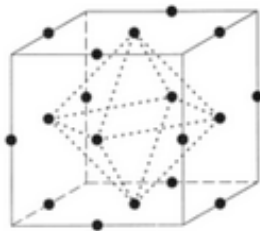
source on right: wayfair.com

Exercise: show that this is a rational-angle line configuration! All of the angles are in fact multiples of one of $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{5}$.

This example is maximal, but it's not so obvious how you would prove it!

Yet another maximal configuration

Consider a cube with vertices $(\pm 1, \pm 1, \pm 1)$. Draw the lines from the center to each of the midpoints of the edges, and to each of the centers of the faces; there are $(12 + 6)/2 = 9$ distinct lines in this configuration (which shows up in representation theory as the B_3 **root system**).



source: Wikimedia Commons

Exercise: show that this is a rational-angle line configuration! All of the angles are in fact multiples of one of $\frac{\pi}{3}, \frac{\pi}{4}$.

This example is again maximal, but again this is not easy to prove.

Even more maximal configurations (more exercises!)

Example

There are 5 different 8-line configurations consisting of seven central diagonals of a regular 60-gon, plus an eighth line not in the same plane.

Example

There are **infinitely many** 6-line configurations of this form. Take two perpendicular lines L_1 and L_2 . Choose a plane containing L_1 but not L_2 , and rotate by $\pm \frac{2\pi}{3}$ around the normal to that plane to get four more lines.

Example

There are **infinitely many** 6-line configurations of this form. Take a “fan” of five lines L_1, L_2, L_3, L_4, L_5 spaced by angles of θ . Then for θ in a suitable range, there is a sixth line perpendicular to L_3 , making angles of $\frac{\pi}{3}$ with L_2 and L_4 , and making angles of θ with L_1 and L_5 .

A classification theorem

Theorem (Kedlaya–Kolpakov–Poonen–Rubinstein, 2020)

The maximal rational-angle line configurations are classified as in the following table.

n	number of maximal rational-angle n -line configurations
\aleph_0	1
15	1
9	1
8	5
6	22, plus 5 one-parameter families
5	29, plus 2 one-parameter families
4	228, plus 10 one-parameter families and 2 two-parameter families
3	1 three-parameter family (the trivial one)

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Finding 4-line configurations

The main difficulty is to classify rational-angle 4-line configurations. To find larger ones, we start with each possible set of 4, then repeatedly try to extend it so that every 4-element subset of the result is in the original list.

To find 4-line configurations, we first classify 6-tuples of angles $(\theta_{ij})_{1 \leq i < j \leq 4}$ that satisfy the following condition:

$$\det \begin{pmatrix} 1 & \cos \theta_{12} & \cos \theta_{13} & \cos \theta_{14} \\ \cos \theta_{12} & 1 & \cos \theta_{23} & \cos \theta_{24} \\ \cos \theta_{13} & \cos \theta_{23} & 1 & \cos \theta_{34} \\ \cos \theta_{14} & \cos \theta_{24} & \cos \theta_{34} & 1 \end{pmatrix} = 0.$$

Proof that this condition is necessary: choose unit vectors along the lines L_1, \dots, L_4 and make the 3×4 matrix A with those vectors as the columns. Then A has rank at most 3 and $A^T A$ is the matrix displayed above.

An algebraic translation

For $z_{jk} = e^{i\theta_{jk}}$, the algebraic condition we wrote down becomes

$$\det \begin{pmatrix} 2 & z_{12} + z_{12}^{-1} & z_{13} + z_{13}^{-1} & z_{14} + z_{14}^{-1} \\ z_{12} + z_{12}^{-1} & 2 & z_{23} + z_{23}^{-1} & z_{24} + z_{24}^{-1} \\ z_{13} + z_{13}^{-1} & z_{23} + z_{23}^{-1} & 2 & z_{34} + z_{34}^{-1} \\ z_{14} + z_{14}^{-1} & z_{24} + z_{24}^{-1} & z_{34} + z_{34}^{-1} & 2 \end{pmatrix} = 0.$$

This is a Laurent polynomial in the six variables z_{jk} , which we want to solve in roots of unity. **Crucial point:** one also has a Regge symmetry on the full solution set of this equation:

$$z_{13}, z_{24}, z_{14}, z_{23} \mapsto \frac{s}{z_{13}}, \frac{s}{z_{24}}, \frac{s}{z_{14}}, \frac{s}{z_{23}}, \quad s := \sqrt{z_{13}z_{24}z_{14}z_{23}}.$$

Unfortunately, this Laurent polynomial has 105 terms, so we cannot view it as a “short” additive relation among roots of unity.

A new approach: extra relations

Another approach (Beukers–Smyth): given the initial polynomial, generate new polynomials with the same solutions in roots of unity.

For example, given a Laurent polynomial $f(x, y)$ over \mathbb{Q} , any solution of $f(x, y) = 0$ in roots of unity is also a solution of one of the polynomials

$$f(x, -y), f(-x, y), f(-x, -y), \\ f(x^2, y^2), f(x^2, -y^2), f(-x^2, y^2), f(-x^2, -y^2).$$

Exercise: prove this using what we discussed earlier! **Hint:** write $x = \zeta_N^i$, $y = \zeta_N^j$ for N minimal, then branch on $N \pmod{4}$ and $i, j \pmod{2}$.

This is very practical! However, the natural generalization (Aliev–Smyth) barely works in practice for 3 variables, and not at all for 4+ variables.

Mod 2 cyclotomic relations

One can also classify additive relations among roots of unity modulo** 2.

E.g., if ζ_1, \dots, ζ_6 are six roots of unity that sum to zero mod 2, then:

- they cancel in pairs (up to signs);
- they form two triples, each of the form $\pm\zeta, \pm e^{2\pi i/3}\zeta, \pm e^{4\pi i/3}\zeta$ for some ζ ;
- or they have the form (for some ζ)
 $\pm\zeta e^{2\pi i/3}, \pm\zeta e^{4\pi i/3}, \pm\zeta e^{2\pi i/5}, \pm\zeta e^{4\pi i/5}, \pm\zeta e^{6\pi i/5}, \pm\zeta e^{8\pi i/5}$.

This helps because our determinant reduces mod 2 to a Laurent polynomial with only 12 monomials:

$$z_{12}^2 z_{34}^2 + z_{12}^{-2} z_{34}^{-2} + z_{12}^2 z_{34}^{-2} + z_{12}^{-2} z_{34}^2 + \dots$$

and this is in the range we can handle (following Poonen–Rubinstein).

**To be precise, this “modulo” is interpreted in the ring of algebraic integers.

The strategy

We now classify rational-angle 4-line configurations as follows.

- Do a C computation to find angle solutions with small denominator (up to 420), discarding those in known parametric families. This finds a putative classification **and** provides a key step in the proof.
- Write down all relations among the 12 monomials that persist mod 2.
- For each relation, make a system of equations that imposes these relations plus the vanishing of the original determinant.
- Use Regge symmetries to reduce the number of systems (down to a few hundred).
- Solve these systems using the Beukers–Smyth approach. To save time, for isolated solutions, we only check that their denominators are in the range covered by the C code.
- For the parametric solutions in roots of unity, convert these back into angles to confirm our guesses for the parametric families.