Roots of unity in geometry (and elsewhere)

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* The UC San Diego campus occupies unceded ancestral homelands of the Kumeyaay Nation. The Kumeyaay people continue to have an important and thriving presence in the region.

Contents

Statement of IMO 1990-6

Problem

Prove that there exists a convex 1990*-gon with the following two properties:*

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The key to solving this is to realize that this is **not** a geometry problem!

Additive relations among roots of unity

Let N be a positive integer and set $\zeta_N := e^{2\pi i/N} \in \mathbb{C}$. For each proper divisor d of N and each $i \in \{0, \dots, d-1\}$, we have

$$\zeta_N^i + \zeta_N^{i+d} + \zeta_N^{i+2d} + \cdots + \zeta_N^{i+(N/d-1)d} = 0.$$

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Consequently, if $a_0, \ldots, a_{N-1} \in \mathbb{Q}$ is a d-**periodic** sequence (i.e., a_j depends only on $j \pmod{d}$) for some proper divisor d of N, then $a_0 + a_1\zeta_N + a_2\zeta_N^2 + \cdots + a_{N-1}\zeta_N^{N-1} = 0$.

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Exercise: solve the problem from here! **Hint**:

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- Prove that the dimension of the $\mathbb{Q}$-vector space generated by the known relations is at least $N - \varphi(N)$. **Hint**: the d-periodic sequences generate a space of dimension d; now use inclusion-exclusion.
- Prove that there are no nonzero relations among $1, \zeta_N, \dots, \zeta_N^{\varphi(N)-1}$. See next slide.

Cyclotomic polynomials

Define the N-th **cyclotomic polynomial** as

$$\Phi_N(x) = \prod_{\substack{i=0 \\ \gcd(i,N)=1}}^{N-1} (x - \zeta_N^i) = \prod_{d|N} (x^d - 1)^{\mu(N/d)}.$$

The first formula gives a polynomial of degree $\varphi(N)$ over $\mathbb{C}$; the second shows that the coefficients are in $\mathbb{Z}$ (by comparing power series over $\mathbb{Z}$).

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In fact $\Phi_N(x)$ is **irreducible** over $\mathbb{Q}$, so there is no nonzero $\mathbb{Q}$-relation among $1, \zeta_N, \dots, \zeta_N^{\varphi(N)-1}$. This can be proved using **Gauss's lemma** plus formal properties of polynomials with mod-p coefficients; see next slide.

Irreducibility of cyclotomic polynomials (sketch)

There is a unique monic irreducible factor $f(x)$ of $\Phi_N(x)$ with $f(\zeta_N) = 0$.
Claim: for every prime p not dividing N, the roots of f are closed under $x \mapsto x^p$; this will imply $\deg(f) \geq \varphi(N)$ (why?) and so $f = \Phi_N$.

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Put $g(x) = (x^N - 1)/f(x)$. If my claim fails, then $g(x^p)$ and $f(x)$ have a root in common, so in fact $g(x^p) = f(x)h(x)$ for some polynomial h with ~~rational~~ **integer** coefficients (Gauss's lemma).

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Now reduce everything modulo p, using bars to denote the reduction:

$$\overline{f}(x)\overline{h}(x) = \overline{g}(x^p) = \overline{g}(x)^p. \qquad (!!)$$

This means that $\overline{f}$ and $\overline{g}$ have a common factor, so $\overline{f}\overline{g}$ has a **repeated** factor.

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$$(\overline{f}\,\overline{g})'(x) = \overline{(fg)'(x)} = Nx^{N-1}$$

has no common factor with $\overline{f}\,\overline{g}(x) = \overline{fg}(x) = x^N - 1$.

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- or they have the form $-\zeta e^{2\pi i/3}, -\zeta e^{4\pi i/3}, \zeta e^{2\pi i/5}, \zeta e^{4\pi i/5}, \zeta e^{6\pi i/5}, \zeta e^{8\pi i/5}$ for some ζ.

Exercise: prove this using what we have already discussed! **Hint**:

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Exercise: prove this using what we have already discussed! **Hint**: one approach uses the following lemma.

Lemma (Conway–Jones, 1974)

Let $\zeta_1, \dots, \zeta_n$ be powers of $\zeta_N = e^{2\pi i/N}$ which sum to zero. Assume also:

- *there is no nonempty proper subset S of $\{1, \dots, n\}$ such that $\sum_{i \in S} \zeta_i = 0$ (that is, we have an* **indecomposable** *additive relation);*
- *there is no root of unity ζ such that $\zeta\zeta_1, \dots, \zeta\zeta_n$ all have order dividing d for some proper divisor d of N.*

Then $n \geq 2 + \sum_{p|N}(p-2)$ where p runs over distinct prime factors of N.

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The last two arguments were heavily computer-assisted. One can probably go a bit further, but the complexity seems to be exponential in n.

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Some applications of this classification

- Counting the intersections of diagonals of a regular polygon (Poonen–Rubinstein)
- The orchard problem: choose n points in the plane to maximize the number of lines through exactly 3 points (Green[§]–Tao[¶])
- Classification of Frobenius–Perron dimensions of fusion categories (Calegari[‖]–Morrison–Snyder)
- Algebraic solutions of the Painlevé VI differential equation (Lisovyy–Tykhyy)
- Computing the possible proportions of elements of a finite group which vanish on some irreducible character (Zeng–Yang–Dolfi)

[§] Represented United Kingdom at IMO 1994–1995.

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A tetrahedron is scissors-congruent to a cube of the same volume if and only if the **Dehn invariant** *is zero.*

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For example, the Dehn invariant is nonzero for a regular tetrahedron (Hilbert's 3rd problem); but it is zero for a tetrahedron with rational dihedral angles.

Tetrahedra with rational dihedral angles: classification

In their 1976 paper, Conway–Jones asked for a classification of tetrahedra with rational dihedral angles. Here is the answer! (Besides our preprint, see also this writeup in *Quanta*.)

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Theorem (Kedlaya–Kolpakov–Poonen–Rubinstein, 2020 preprint)

Up to symmetry, any tetrahedron in $\mathbb{R}^3$ *with all dihedral angles rational is either one of* 59 *sporadic examples (next slide) or has one of the forms*

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right) \qquad \text{for } \frac{\pi}{6} < x < \frac{\pi}{2},$$

$$\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right) \qquad \text{for } \frac{\pi}{6} < x \leq \frac{\pi}{3}.$$

Convention: we list dihedral angles in the order $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$, where α_{ij} means the angle between faces i and j.

Sporadic tetrahedra (key on the next slide)

N	$(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ as multiples of π/N
12	$(3, 4, 3, 4, 6, 8) = H_2(\pi/4)$
24	$(5, 9, 6, 8, 13, 15)$
12	$(3, 6, 4, 6, 4, 6) = T_0$
24	$(7, 11, 7, 13, 8, 12)$
15	$(3, 3, 3, 5, 10, 10) = T_{18}$, $(2, 4, 4, 4, 10, 10)$, $(3, 3, 4, 4, 9, 11)$
15	$(3, 3, 5, 5, 9, 9) = T_7$
15	$(5, 5, 5, 9, 6, 6) = T_{23}$, $(3, 7, 6, 6, 7, 7)$, $(4, 8, 5, 5, 7, 7)$
21	$(3, 9, 7, 7, 12, 12)$, $(4, 10, 6, 6, 12, 12)$, $(6, 6, 7, 7, 9, 15)$
30	$(6, 12, 10, 15, 10, 20) = T_{17}$, $(4, 14, 10, 15, 12, 18)$
60	$(8, 28, 19, 31, 25, 35)$, $(12, 24, 15, 35, 25, 35)$, $(13, 23, 15, 35, 24, 36)$, $(13, 23, 19, 31, 20, 40)$
30	$(6, 18, 10, 10, 15, 15) = T_{13}$, $(4, 16, 12, 12, 15, 15)$, $(9, 21, 10, 10, 12, 12)$
30	$(6, 6, 10, 12, 15, 20) = T_{16}$, $(5, 7, 11, 11, 15, 20)$
60	$(7, 17, 20, 24, 35, 35)$, $(7, 17, 22, 22, 33, 37)$, $(10, 14, 17, 27, 35, 35)$, $(12, 12, 17, 27, 33, 37)$
30	$(6, 10, 10, 15, 12, 18) = T_{21}$, $(5, 11, 10, 15, 13, 17)$
60	$(10, 22, 21, 29, 25, 35)$, $(11, 21, 19, 31, 26, 34)$, $(11, 21, 21, 29, 24, 36)$, $(12, 20, 19, 31, 25, 35)$
30	$(6, 10, 6, 10, 15, 24) = T_6$
60	$(7, 25, 12, 20, 35, 43)$
30	$(6, 12, 6, 12, 15, 20) = T_2$
60	$(12, 24, 13, 23, 29, 41)$
30	$(6, 12, 10, 10, 15, 18) = T_3$, $(7, 13, 9, 9, 15, 18)$
60	$(12, 24, 17, 23, 33, 33)$, $(14, 26, 15, 21, 33, 33)$, $(15, 21, 20, 20, 27, 39)$, $(17, 23, 18, 18, 27, 39)$
30	$(6, 15, 6, 18, 10, 20) = T_4$, $(6, 15, 7, 17, 9, 21)$
60	$(9, 33, 14, 34, 21, 39)$, $(9, 33, 15, 33, 20, 40)$, $(11, 31, 12, 36, 21, 39)$, $(11, 31, 15, 33, 18, 42)$
30	$(6, 15, 10, 15, 12, 15) = T_1$, $(6, 15, 11, 14, 11, 16)$, $(8, 13, 8, 17, 12, 15)$, $(8, 13, 9, 18, 11, 14)$, $(8, 17, 9, 12, 11, 16)$, $(9, 12, 9, 18, 10, 15)$
30	$(10, 12, 10, 12, 15, 12) = T_5$
60	$(19, 25, 20, 24, 29, 25)$

How to read the table

Each tetrahedron is represented by an integer N and a list of six integers, representing $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$ as multiples of $\frac{\pi}{N}$.

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The groups between horizontal lines are orbits for a certain "extra" symmetry group (more on this shortly).

Regge symmetry

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For any tetrahedron with edges $(\ell_{12}, \ell_{34}, \ell_{13}, \ell_{24}, \ell_{14}, \ell_{23})$ *and dihedral angles* $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$*, there is another with edges*

$$(\ell_{12}, \ell_{34}, s - \ell_{13}, s - \ell_{24}, s - \ell_{14}, s - \ell_{23}), \; s = \frac{1}{2}(\ell_{13} + \ell_{24} + \ell_{14} + \ell_{23})$$

and dihedral angles

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In 1999, Roberts observed that the Dehn invariant is also preserved. In 2019, Akopyan–Izmestiev gave a “classical” proof of the theorem.

Consequences of Regge symmetry

The family of tetrahedra with dihedral angles $\left(\frac{\pi}{2},\ \frac{\pi}{2},\ \pi - 2x,\ \frac{\pi}{3},\ x,\ x\right)$ was discovered by Hill in 1895. Applying a Regge symmetry gives the family $\left(\frac{5\pi}{6} - x,\ \frac{\pi}{6} + x,\ \frac{2\pi}{3} - x,\ \frac{2\pi}{3} - x,\ x,\ x\right)$.

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Together with the action of S_4 on faces, the Regge symmetry generates a larger group acting on isomorphism classes of tetrahedra. Our table of sporadic tetrahedra indicates orbits for this larger group.

In particular, all of the sporadic examples are “explained” by the classical ones via this larger symmetry group **except** for the ones with $N = 21$.

Rational-angle line configurations

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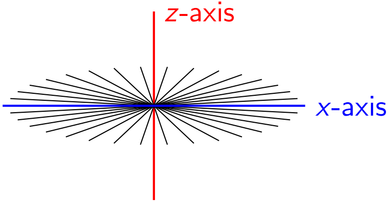
For example, the three coordinate axes form angles of $\frac{\pi}{2}$, but this is not maximal...

A maximal configuration

Consider all of the lines in the *xy*-plane that form a rational angle with the *x*-axis, together with the *z*-axis. This is a rational-angle line configuration.



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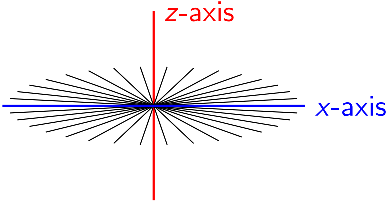

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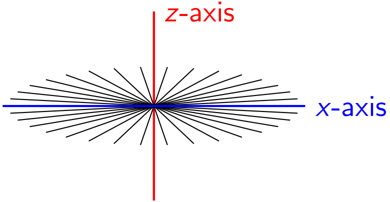
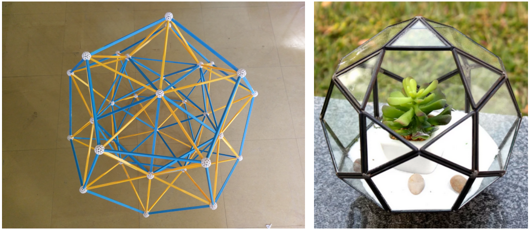


Exercise: show that this is in fact maximal.

We will see this is the only **infinite** maximal configuration.

Another maximal configuration

Consider a dodecahedron, and draw the 15 lines from the center to the midpoints of each of the 30 edges. (These are also the midpoints of the edges of an icosahedron, or the vertices of an **icosidodecahedron**.)



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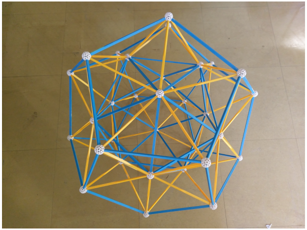


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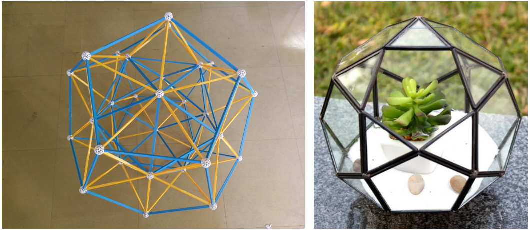

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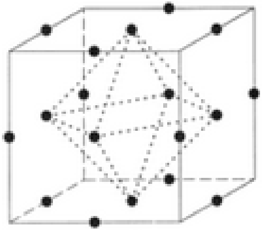
Exercise: show that this is a rational-angle line configuration! All of the angles are in fact multiples of one of $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}$.

This example is maximal, but it's not so obvious how you would prove it!

Yet another maximal configuration

Consider a cube with vertices $(\pm 1, \pm 1, \pm 1)$. Draw the lines from the center to each of the midpoints of the edges, and to each of the centers of the faces; there are $(12+6)/2 = 9$ distinct lines in this configuration (which shows up in representation theory as the B_3 **root system**).



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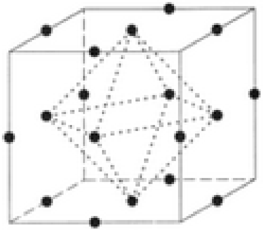

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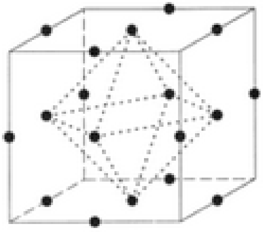

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Exercise: show that this is a rational-angle line configuration! All of the angles are in fact multiples of one of $\frac{\pi}{3}, \frac{\pi}{4}$.

This example is again maximal, but again this is not easy to prove.

Even more maximal configurations (more exercises!)

Example

There are 5 different 8-line configurations consisting of seven central diagonals of a regular 60-gon, plus an eighth line not in the same plane.

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Example

There are **infinitely many** 6-line configurations of this form. Take a "fan" of five lines L_1, L_2, L_3, L_4, L_5 spaced by angles of θ. Then for θ in a suitable range, there is a sixth line perpendicular to L_3, making angles of $\frac{\pi}{3}$ with L_2 and L_4, and making angles of θ with L_1 and L_5.

A classification theorem

Theorem (Kedlaya–Kolpakov–Poonen–Rubinstein, 2020)

The maximal rational-angle line configurations are classified as in the following table.

n	number of maximal rational-angle n-line configurations
$\aleph_0$	1
15	1
9	1
8	5
6	22, plus 5 one-parameter families
5	29, plus 2 one-parameter families
4	228, plus 10 one-parameter families and 2 two-parameter families
3	1 three-parameter family (the trivial one)

Contents

Finding 4-line configurations

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To find 4-line configurations, we first classify 6-tuples of angles $(\theta_{ij})_{1 \leq i < j \leq 4}$ that satisfy the following condition:

$$\det \begin{pmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{12} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{13} & \cos\theta_{23} & 1 & \cos\theta_{34} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & 1 \end{pmatrix} = 0.$$

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Proof that this condition is necessary: choose unit vectors along the lines $L_1, \dots, L_4$ and make the 3×4 matrix A with those vectors as the columns. Then A has rank at most 3 and $A^T A$ is the matrix displayed above.

An algebraic translation

For $z_{jk} = e^{i\theta_{jk}}$, the algebraic condition we wrote down becomes

$$\det \begin{pmatrix} 2 & z_{12} + z_{12}^{-1} & z_{13} + z_{13}^{-1} & z_{14} + z_{14}^{-1} \\ z_{12} + z_{12}^{-1} & 2 & z_{23} + z_{23}^{-1} & z_{24} + z_{24}^{-1} \\ z_{13} + z_{13}^{-1} & z_{23} + z_{23}^{-1} & 2 & z_{34} + z_{34}^{-1} \\ z_{14} + z_{14}^{-1} & z_{24} + z_{24}^{-1} & z_{34} + z_{34}^{-1} & 2 \end{pmatrix} = 0.$$

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This is a Laurent polynomial in the six variables z_{jk}, which we want to solve in roots of unity. **Crucial point**: one also has a Regge symmetry on the full solution set of this equation:

$$z_{13}, z_{24}, z_{14}, z_{23} \mapsto \frac{s}{z_{13}}, \frac{s}{z_{24}}, \frac{s}{z_{14}}, \frac{s}{z_{23}}, \qquad s := \sqrt{z_{13}z_{24}z_{14}z_{23}}.$$

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Unfortunately, this Laurent polynomial has 105 terms, so we cannot view it as a "short" additive relation among roots of unity.

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Exercise: prove this using what we discussed earlier! **Hint**: write $x = \zeta_N^i$, $y = \zeta_N^j$ for N minimal, then branch on $N \pmod 4$ and $i, j \pmod 2$.

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This is very practical! However, the natural generalization (Aliev–Smyth) barely works in practice for 3 variables, and not at all for 4+ variables.

Mod 2 cyclotomic relations

One can also classify additive relations among roots of unity modulo** 2.
E.g., if $\zeta_1, \ldots, \zeta_6$ are six roots of unity that sum to zero mod 2, then:

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This helps because our determinant reduces mod 2 to a Laurent polynomial with only 12 monomials:

$$z_{12}^2 z_{34}^2 + z_{12}^{-2} z_{34}^{-2} + z_{12}^2 z_{34}^{-2} + z_{12}^{-2} z_{34}^2 + \cdots$$

and this is in the range we can handle (following Poonen–Rubinstein).

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- For the parametric solutions in roots of unity, convert these back into angles to confirm our guesses for the parametric families.