The Robbins phenomenon: Cluster algebras and p-adic arithmetic

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Computing with p-adic numbers

Recall that the field $\mathbb{Q}_p$ of *p-adic numbers* is obtained by completing $\mathbb{Q}$ for the p-adic valuation v_p, defined by

$$v_p\left(p^e \frac{r}{s}\right) = e \qquad (e, r, s \in \mathbb{Z}; \gcd(r, p) = \gcd(s, p) = 1).$$

One may informally think of elements of $\mathbb{Q}_p$ as base-p expansions which continue infinitely far to the left. Just like in $\mathbb{R}$, arbitrary elements of $\mathbb{Q}_p$ are not finitary and so cannot be represented exactly on a computer.

There are several possible schemes for systematically approximating p-adic numbers with exact rational numbers. The one we consider in this talk is the p-adic analogue of *floating-point (FP) arithmetic*.

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p-adic floating-point: a computational model

Fix a positive integer b (the *working relative precision*). We approximate $x \in \mathbb{Q}_p$ by rational numbers of the form $p^e m$ where e is an integer (the *exponent*) and m is an integer in $\{0, \dots, p^b - 1\}$ not divisible by p (the *mantissa*). Some special case behavior is required to deal with 0.

The *relative accuracy* of an approximation $p^e m$ to x is

$$\max\{0, v_p\left(\frac{x}{p^e m} - 1\right)\}.$$

counting the number of correct p-adic digits of the mantissa starting from the right. For instance, here are the accuracies of some approximations of $-1 = \cdots 1111111_2$ with $p = 2$ and $b = 7$:

$2^0 \cdot 1010111_2$	accuracy 3
$2^0 \cdot 1010101_2$	accuracy 1
$2^0 \cdot 1011100_2$	invalid approximation (last digit should be nonzero)
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Arithmetic in p-adic working relative precision b

Consider $x, y \in \mathbb{Q}_p$ equipped with approximations $p^{e_1} m_1, p^{e_2} m_2$ with relative accuracy each at least d.

- Write $m_1 m_2 \equiv m_3 \pmod{p^b}$ where m_3 is a p-adic unit. Then $x \times y$ is approximated by $p^{e_1+e_2} m_3$ with relative accuracy $\geq d$. This operation is thus "exact." Similarly for $x \div y$.
- If $e_1 < e_2$, we write $m_1 + p^{e_2-e_1} m_2 \equiv m_3 \pmod{p^b}$ where m_3 is a p-adic unit. Then $x + y$ is approximated by $p^{e_1} m_3$ with relative accuracy $\geq d$, so this operation is also exact. Similarly if $e_1 > e_2$.
- If $e_1 = e_2$, we write $m_1 + m_2 \equiv p^{e_3} m_3 \pmod{p^b}$ where m_3 is a p-adic unit. Then $x + y$ is approximated by $p^{e_1+e_3} m_3$ with relative accuracy $\geq d - e_3$. If $e_3 > 0$, this creates a loss of accuracy (and the operation is multivalued). An illustration with $p = 3$:

$$
\begin{array}{ccccccc}
 & \cdots & ? & 0 & 0 & 1 & 1_{(3)} & \times 3^2 \\
+ & \cdots & ? & 0 & 0 & 0 & 2_{(3)} & \times 3^2 \\
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Numerical stability in p-adic arithmetic

The most common treatment of p-adic arithmetic (in *Pari*, *Sage*, *Magma*, etc.) is to carry along the relative accuracy of each approximation as an extra datum (compare with *interval arithmetic* over $\mathbb{R}$). A typical sequence of operations in p-adic FP arithmetic then experiences progressive loss of relative accuracy over the course of the computation. Avoiding this amounts to a p-adic version of the subject of *numerical stability*.

In this talk, we describe some computations for which term-by-term analysis of relative accuracy (as on the previous slide) turns out to be much too pessimistic.

These involve new theorems and conjectures about cluster algebras!

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p-adic floating-point: an algebraic model

To analyze our floating-point model, it will be useful to give an algebraic reformulation. Write out an exact computation in $\mathbb{Q}_p$ as a straight-line program, then replace each step

$$x_n := y * z \qquad (* \in \{+, -, \times, \div\})$$

with

$$x_n := (1+\varepsilon_1)y * (1+\varepsilon_2)z$$

for some arbitrary $\varepsilon_i \in p^b\mathbb{Z}_p$. If x is the intended answer and $\tilde{x}$ the computed answer, then the *absolute accuracy* of the computation is the minimum of $v_p(x-\tilde{x})$ over all choices of the ε factors.

This model allows us to estimate errors by working formally over rings of indeterminates corresponding to inputs of the computation and to ε factors. (It also accounts for corner cases, e.g., underflow to 0.)

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The Somos-4 recurrence

The Somos-4 recurrence relation

$$x_{n+4} = \frac{x_{n+1}x_{n+3} + x_{n+2}^2}{x_n}$$

exhibits the *Laurent phenomenon*:

$$x_n \in \mathbb{Z}[x_0^{\pm}, x_1^{\pm}, x_2^{\pm}, x_3^{\pm}] \qquad (n = 0, 1, \dots).$$

One way to see this is to identify the x_n as cluster variables for an infinite sequence of seed mutations of the following form:

$$\begin{tikzcd}
n \arrow[dr, Rightarrow] & n+1 \arrow[l] \arrow[dl, Rightarrow] \\
n+3 \arrow[u] \arrow[r] & n+2 \arrow[u, Rrightarrow]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
n+4 \arrow[r] \arrow[d] & n+1 \arrow[dl, Rightarrow] \\
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Somos-4 and p-adic floating-point

If we start with $x_0, \ldots, x_3$ in $\mathbb{Z}_p^\times$, then the Laurent phenomenon implies that $x_n \in \mathbb{Z}_p$ for all $n \geq 0$. Let us try to compute $x_0, \ldots, x_m$ using p-adic FP arithmetic with working relative precision b; what we end up with is a sequence of approximations $X_0, \ldots, X_m$.

A loss of relative accuracy occurs in computing X_{n+4} whenever

$$v_p(X_{n+1}X_{n+3} + X_{n+2}^2) > v_p(X_{n+1}X_{n+3}) = v_p(X_{n+2}^2).$$

The amount of loss is $v_p(X_{n+1}X_{n+3} + X_{n+2}^2) - v_p(X_{n+1}X_{n+3})$, which is bounded above by $v_p(X_{n+4}) + v_p(X_n)$ provided that $X_{n+1}, X_{n+3} \in \mathbb{Z}_p$.

Consequently, if $X_0, \ldots, X_m \in \mathbb{Z}_p$, then

$$v_p(x_m - X_m) \geq b - 2\sum_{n=0}^{m} v_p(X_n).$$

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The numerical Robbins phenomenon for Somos-4

Surprisingly, the previous bound is far from optimal! In some sense, the errors arising from the $v_p(X_n)$ do not compound after all.

Theorem (Numerical Robbins phenomenon for Somos-4)

For x_n, X_n as on the previous slide,

$$v_p(x_m - X_m) \geq b - r, \qquad r = \max\{v_p(X_n) : n = 0, \dots, m-4\}.$$

Via the algebraic interpretation of FP arithmetic, one can deduce this theorem (and a corresponding statement with $\mathbb{Z}_p$ replaced by any DVR) from a statement about a certain algebraic deformation of the Somos-4 recurrence, which we describe next.

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To study Somos-4 in the algebraic FP model, we simulate the computation of $x_0, \ldots, x_m$ by introducing a second sequence $X_0, \ldots, X_m$ with $X_i = x_i$ for $i = 0, 1, 2, 3$ and

$$X_{n+4} = \frac{X_{n+1}X_{n+3}(1+\varepsilon_n^+) + X_{n+2}^2(1+\varepsilon_n^-)}{X_n}$$

(using the group structure of $1 + p^b\mathbb{Z}_p$ to consolidate ε factors).

Theorem (Robbins phenomenon for Somos-4)

Within the field $\mathrm{Frac}(\mathbb{Z}[X_n, \varepsilon_n^{\pm}])$*, we have*

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we may specialize to $\mathbb{Q}_p$ in two different ways.

- If we specialize X_i to x_i for $i = 0,1,2,3$ and $\varepsilon_j^\pm$ to 0, then X_m specializes to x_m, the true value computed by Somos-4.
- If we instead specialize the $\varepsilon_j^\pm$ to elements of $p^b\mathbb{Z}_p$, then X_m specializes to x_m plus a polynomial over $\mathbb{Z}_p$ in the images of the $\varepsilon_j^\pm/X_k$, each of which has valuation at least $b-r$.

This yields the numerical Robbins phenomenon for Somos-4:

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We conjecture that the previous theorems remain true if Somos-4 is replaced by any sequence of cluster variables corresponding to a sequence of seed mutations. This is what we call the *Robbins phenomenon* in its numerical and algebraic forms. (A formal statement will be given later.)

This represents a generalization of a conjecture of David Robbins, who in 2005 formulated a special case of this conjecture: the numerical Robbins phenomenon for the *Dodgson condensation* recurrence.

Robbins did not consider any recurrences other than condensation, and previous study of his conjecture has centered on direct algebraic manipulations. Our hope is that insights from the general theory of cluster algebras (driven by such small examples as Somos-4) may shed light on Robbins's conjecture; it has already led us to a proof of a qualitative version of the conjecture.

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Robbins did not consider any recurrences other than condensation, and previous study of his conjecture has centered on direct algebraic manipulations. Our hope is that insights from the general theory of cluster algebras (driven by such small examples as Somos-4) may shed light on Robbins's conjecture; it has already led us to a proof of a qualitative version of the conjecture.

Condensation of determinants, being a new and brief method for computing their arithmetical values

In 1866, C.L. Dodgson (a/k/a Lewis Carroll) proposed a procedure for computing the determinant of an $n \times n$ matrix A. This procedure is to compute the matrix $A[k]$ of connected k-minors of A for $k = 0, \ldots, n$, using the three-term recurrence provided by Jacobi's identity

$$D \cdot C = NW \cdot SE - NE \cdot SW.$$

In this identity, D is the determinant of a $k \times k$ matrix, C is the central $(k-2)$-minor (with the convention that any 0-minor equals 1), and the other terms are the $(k-1)$-minors occurring in the indicated cardinal directions.

But there is a catch: it can happen that $C = 0$, in which case Jacobi's identity provides no way to solve for D.

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Lifting of determinants

Robbins was interested in using condensation to compute determinants over $\mathbb{F}_p$. He noticed that the problem of zero denominators could be finessed by lifting to $\mathbb{Z}$, which would convert most zeroes into nonzero quantities; however, computing over $\mathbb{Z}$ is expensive due to coefficient explosion.

Instead, he proposed to work over $\mathbb{Q}_p$, using p-adic FP arithmetic to limit the bitsize of matrix entries. To get the desired determinant over $\mathbb{F}_p$, he needed a way to certify that a given floating-point calculation yields an answer correct to at least *one* digit of accuracy.

He thus set about conducting numerical experiments, expecting to observe the usual compounding losses of accuracy associated with p-adic FP computations. To his surprise, these did not occur!

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The conjecture of Robbins

Conjecture (Robbins, 2005)

Let A be an $n \times n$ matrix over $\mathbb{Z}_p$. For $k = 0, \dots, n$, compute a square matrix $\tilde{A}_k$ of size $n - k + 1$ using p-adic FP arithmetic with working relative precision b, and put $D = \tilde{A}_{n;1,1}$ and

$$r = \max\{v_p(\tilde{A}_{k;i,j}) : 0 < k < n;\ 1 < i,j < n-k\}.$$

Then $v(\det(A) - D) \geq b - r$.

Note that we do not require the entries of A to be p-adic units, in contrast with the Somos-4 case. This makes some sense: the Laurent phenomenon here is the fact that any minor is an ordinary polynomial (not just a Laurent polynomial) in the matrix entries.

For an algebraic statement which implies this conjecture, see next slide.

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An algebraic refinement of Robbins's conjecture

Conjecture (Robbins phenomenon for condensation, algebraic form)

Put $R = \mathbb{Z}[A_{i,j} : i, j = 1, \dots, n]$. *For* $k = 0, \dots, n$, *define the square matrix* $\tilde{A}_k$ *of size* $n - k + 1$ *by setting*

$$\tilde{A}_{0;i,j} = 1, \qquad \tilde{A}_{1;i,j} = A_{i,j}, \qquad \textit{and}$$

$$\tilde{A}_{k+1;i,j} = \frac{(1 + \varepsilon^{+}_{k;i,j})\tilde{A}_{k;i,j}\tilde{A}_{k;i+1,j+1} - (1 + \varepsilon^{-}_{k;i,j})\tilde{A}_{k;i+1,j}\tilde{A}_{k;i,j+1}}{\tilde{A}_{k-1;i+1,j+1}}.$$

Then within the field $\mathrm{Frac}(\mathbb{Z}[\tilde{A}_{k;i,j}, \varepsilon^{\pm}_{k;i,j}])$, *we have*

$$\tilde{A}_{n;1,1} \in R\left[\frac{\varepsilon^{\pm}_{k;i,j}}{\tilde{A}_{k';i',j'}} : \begin{array}{c} 1 \leq k \leq n-1; \quad 1 \leq i, j \leq n-k; \\ (k' = k < n-1 \textit{ and } i'-i, j'-j \in \{0,1\}) \\ \textit{or } (k', i', j') = (k-2, i+1, j+1) \end{array}\right].$$

Contents

Sequences from cluster algebras

Both the Somos-4 recurrence and the Dodgson condensation recurrence consist of cluster variables corresponding to certain sequences of seed mutations. This suggests formulating the numerical and algebraic Robbins phenomena in this level of generality, as follows.

Start with a skew-symmetrizable seed B_0 of rank k, and let $t_1, \dots, t_k$ be cluster variables at B_0 over some base ring R_0.

Perform a sequence of seed mutations to obtain new seeds $B_1, \dots, B_m$. For $i = 1, \dots, m$, let x_i be the new cluster variable introduced by the mutation from B_{i-1} to B_i; it is defined by an exchange relation of the form

$$x_i = \frac{u^+ \mu_i^+ + u^- \mu_i^-}{d_i}$$

where d_i is a cluster variable at B_{i-1}, μ_i^+ and μ_i^- are monomials in the other cluster variables at B_{i-1}, and u^+, u^- are units in R_0.

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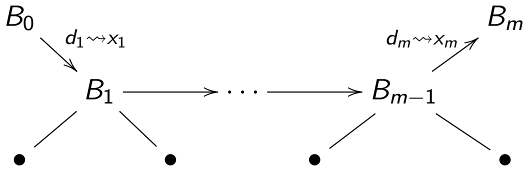
An illustration

We may represent the sequence of seed mutations as a caterpillar graph:



Reminders:

- The cluster variables at each B_i are a subset of $t_1, \dots, t_k, x_1, \dots, x_i$.
- Therefore, μ_i^+, μ_i^- are monomials in $t_1, \dots, t_k, x_1, \dots, x_{i-1}$.
- Similarly, d_i is one of $t_1, \dots, t_k, x_1, \dots, x_{i-1}$.

Numerical Robbins phenomenon for cluster algebras

Let us specialize $t_1, \dots, t_k$ to $\mathbb{Z}_p^\times$, so that $x_1, \dots, x_m$ specialize to elements of $\mathbb{Z}_p$ (ignoring zero divisions). Let us now emulate the computation of $x_1, \dots, x_m$ using p-adic FP arithmetic, obtaining the sequence $X_1, \dots, X_m$. Let D_i be the element of the sequence $t_1, \dots, t_k, X_1, \dots, X_{i-1}$ corresponding to d_i.

Conjecture (Numerical Robbins phenomenon for cluster algebras)

We have

$$v_p(X_m - x_m) \geq b - r, \qquad r = \max\{v_p(D_i) : i = 1, \dots, m\}.$$

Yet again, it is natural to propose an algebraic analogue which implies this conjecture.

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Robbins phenomenon for cluster algebras

Conjecture (Robbins phenomenon for cluster algebras)

Put $R = R_0[t_1^{\pm}, \dots, t_k^{\pm}]$. View d_i, μ_i^+, μ_i^- as polynomials in $t_1, \dots, t_k, x_1, \dots, x_{i-1}$. Define $X_1, \dots, X_m$ by

$$X_i = \frac{u_i^+ M_i^+ (1 + \varepsilon_i^+) + u_i^- M_i^- (1 + \varepsilon_i^-)}{D_i}$$

with D_i, M_i^+, M_i^- the evaluations of d_i, μ_i^+, μ_i^- at $t_1, \dots, t_k, X_1, \dots, X_{i-1}$. Then

$$X_m \in R\left[\frac{\varepsilon_i^{\pm}}{D_\alpha} : i = 1, \dots, m,\ D_\alpha | D_i M_i^+ M_i^-\right].$$

A natural first step would be to check this conjecture "to first order", i.e., modulo the square of the ideal generated by the ε factors.

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From cluster algebras to Dodgson condensation

Recall that the Laurent phenomenon for Dodgson condensation is stronger than what follows by interpreting the condensation recurrence in terms of cluster algebras, because one gets ordinary polynomials rather than Laurent polynomials in the matrix entries.

One can recover the stronger statement for condensation by adjoining a power series variable T, building a matrix A_{-1} of all 1s, then building A_0 from top left to bottom right so that $A_{0;i,j} \equiv 1 \pmod{T}$ and

$$A_{0;i,j}A_{0;i+1,j+1} - A_{0;i+1,j}A_{0;i,j+1} = TA_{1;i,j}.$$

(Modulo T^2, the left side is $A_{0;i+1,j+1} - 1$ plus given multiples of T.) Then $A_{n;1,1} \equiv T^n \pmod{T^{n+1}}$.

By a similar method, one may deduce the Robbins phenomenon for condensation from the Robbins phenomenon for cluster algebras.

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A weak version of the Robbins phenomenon

Theorem (Weak Robbins phenomenon for cluster algebras)

With notation as in the previous conjecture,

$$X_m \in R\left[\frac{\varepsilon_i^+}{\prod\{D_\alpha : D_\alpha | D_i M_i^-\}}, \frac{\varepsilon_i^-}{\prod\{D_\alpha : D_\alpha | D_i M_i^+\}} : i = 1, \dots, m\right].$$

Corollary (Weak numerical Robbins for cluster algebras)

Set notation as in the numerical Robbins conjecture for cluster algebras. Let c be the maximum number of cluster variables appearing in any one of the $\mu_i^\pm$ (so $c \leq k-1$). Then

$$v_p(X_m - x_m) \geq b - (c+1)r, \qquad r = \max\{v_p(D_i) : i = 1, \dots, m\}.$$

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The caterpillar lemma with deformations

The weak Robbins phenomenon is proved using a form of the *caterpillar lemma*, which we prefer to state using *upper bounds* (emulating "Cluster algebras 3"). For B a seed, let R_B be the Laurent polynomial ring over R_0 in the cluster variables at B. Put $U(B) = R_B \cap \bigcap R_{B'}$ for B' running over all seeds obtained from B by a single mutation.

Theorem (Caterpillar lemma, upper bounds formulation)

Assume principal coefficients. For any skew-symmetrizable seeds B and B' related by mutations, there is a natural isomorphism $U(B) \cong U(B')$.

To obtain weak Robbins, adjoin power series variables δ_i and deform the exchange relations to

$$x_i = \frac{u_i^+ \mu_i^+ + u_i^- \mu_i^- + \delta_i \prod\{d_\alpha : d_\alpha | d_i \mu_i^+ \mu_i^-\}}{d_i}.$$

The proof of the caterpillar lemma carries over.

The Robbins phenomenon in some special cases

In certain cases, in the ring $R = R_0[t_1^{\pm}, \dots, t_k^{\pm}]$ any two cluster variables at the same seed generate the unit ideal. In such cases, the weak algebraic Robbins theorem implies the full algebraic Robbins conjecture (and a strong version of numerical Robbins using *fixed-point arithmetic*).

This is mostly limited to cases in which every mutation in the sequence involves all of the cluster variables. Familiar examples of this include:

- Somos-4;
- Somos-5: $x_{n+5} = \dfrac{x_{n+1}x_{n+4} + x_{n+2}x_{n+3}}{x_n}$;
- the Markoff recurrence: $(x, y, z) \to \left(x, y, \dfrac{x^2 + y^2}{z}\right)$.

The Markoff recurrence serves as a typical counterexample against overly optimistic statements about cluster algebras (e.g., finite generation), so it is reassuring to see it in this list!

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Consequences for Dodgson condensation

For Dodgson condensation, we deduce the following weak version of the numerical Robbins phenomenon.

Theorem (Weak numerical Robbins phenomenon for condensation)

Let A be an $n \times n$ matrix over $\mathbb{Z}_p$. For $k = 0, \dots, n$, compute a square matrix $\tilde{A}_k$ of size $n-k+1$ using p-adic FP arithmetic with working relative precision b, and put $D = \tilde{A}_{n;1,1}$ and

$$r = \max\{v_p(\tilde{A}_{k:i,j}) : 0 < k < n; 1 < i, j < n-k\}.$$

Then $v(\det(A) - D) \geq b - 3r$.

Recall that the Robbins conjecture is that $v(\det(A) - D) \geq b - r$. It might be possible to get closer by understanding the valuation profiles of certain sets of cluster variables (i.e., images of tropicalizations).

Beyond cluster algebras

One might guess that *any* recurrence exhibiting the Laurent phenomenon, or perhaps just any recurrence susceptible to the caterpillar lemma, exhibits the numerical Robbins phenomenon. This is false; one can find explicit counterexamples using the Somos-6 recurrence

$$x_{n+6} = \frac{x_{n+1}x_{n+5} + x_{n+2}x_{n+4} + x_{n+3}^2}{x_n}.$$

However, a qualitative version of the phenomenon does persist in all such cases. For Somos-6, for FP approximations $X_0, \dots, X_m$ computed with relative precision b, for $r = \max\{v_p(X_n) : n = 0, \dots, m-6\}$, we prove

$$v_p(x_m - X_m) \geq b - 4r$$

using our modified caterpillar lemma, and numerical evidence suggests that

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