Absolute de Rham cohomology? A fantasy in the key of p

Kiran S. Kedlaya

Department of Mathematics, Massachusetts Institute of Technology; kedlaya@mit.edu
Department of Mathematics, University of California, San Diego

Witt vectors, foliations, and absolute de Rham cohomology
名古屋大学 (Nagoya University), November 22, 2010

For slides, see http://math.mit.edu/~kedlaya/papers/talks.shtml.

Rated W for Witt vectors, wild speculation, and general weirdness.

Supported by NSF, DARPA, MIT, UCSD.

Contents

Contents

Zeta functions and L-functions

For X a scheme of finite type over $\mathbb{Z}$, the *zeta function* of X is defined in the halfplane $\mathrm{Re}(s) > \dim(X)$ as the absolutely convergent Dirichlet series

$$\zeta(X, s) = \prod_{x \in X \text{ closed}} (1 - \#(\kappa_x)^{-s})^{-1}.$$

Many famous questions surround this function.

- Does it admit meromorphic continuation over $\mathbb{C}$?
- Where do the poles and zeroes occur?
- What is the arithmetic meaning of the values $\zeta(X, s)$ when $s \in \mathbb{Z}$?

One can factor $\zeta(X, s)$ into L-functions corresponding to *motives* comprising X, and the same questions apply. E.g., for X the ring of integers in a number field, $\zeta(X, s)$ factors as the Riemann zeta function times *Artin L-functions*.

Weil cohomologies and spectral interpretations

For X of finite type over $\mathbb{F}_q$, a *Weil cohomology* theory, mapping X to certain vector spaces $H^i(X)$ over a field of characteristic zero, provides a *spectral interpretation* of $\zeta(X, s)$ via the formula

$$\zeta(X, s) = \prod_i \det(1 - q^{-s}\mathrm{Frob}_q, H^i(X))^{(-1)^{i+1}}.$$

Existence of a Weil cohomology theory immediately implies analytic continuation for L-functions of pure motives comprising X. The Riemann hypothesis and the interpretation of special values lie deeper.

A familiar example is *étale cohomology* with values in $\mathbb{Q}_\ell$ for any given $\ell \neq p$. A possibly less familiar example is *rigid cohomology*, taking values in a suitable p-adic field; this is a form of *de Rham cohomology* in positive characteristic. More on this in the next two slides.

p-adic Weil cohomology: origins

Dwork's original proof of the rationality of zeta functions of varieties over finite fields used a p-adic analytic trace formula, without a cohomological interpretation. Some links were found to differential forms (Katz).

Monsky and Washnitzer introduced *formal cohomology* for smooth affine varieties over positive characteristic fields. The idea: take a *weakly complete lift* in characteristic 0 and compute algebraic de Rham cohomology. The lift is functorial *up to homotopy* (thanks to the completion) and has well-behaved cohomology (thanks to weakness).

Based on Grothendieck's site-theoretic description of algebraic de Rham cohomology, Berthelot developed *crystalline cohomology* for smooth proper schemes over positive characteristic fields. This is again based on local lifting, in the form of *infinitesimal thickenings with divided powers*.

Rigid cohomology

Berthelot's *rigid cohomology* takes values in $K = W(\mathbb{F}_q)[p^{-1}]$. Given X, one can locally embed X into some Y which lifts nicely (e.g., projective space), cut out the *tube* in the generic fibre of the lift of Y, then compute de Rham cohomology on a strict neighborhood. This description can be made functorial using le Stum's *overconvergent site*, and extends to algebraic stacks (Brown).

It is true but nontrivial that this gives *finite-dimensional* vector spaces over K. In fact, one has a good theory of coefficient objects resembling algebraic $\mathcal{D}$-modules (Berthelot, Caro, Chiarellotto, Crew, Kedlaya, Shiho, Tsuzuki, etc.)

Contents

A modest proposal

Can one describe an *absolute cohomology* theory for schemes of finite type over $\mathbb{Z}$, proving a spectral interpretation of $\zeta(X, s)$? The correct form of this question is suggested by:

- trace formulas for ζ and the like (Weil);
- analogies with phenomena appearing for *foliated spaces* (Deninger).

The latter may (should?) be viewed as *noncommutative spaces* (Connes).

Bonus question: does a related construction produce *p-adic L-functions*? These are only known in a few cases, where they are obtained by interpolation of archimedean special values.

Double bonus question: can one explain special values this way?

Triple bonus question: what about the Riemann hypothesis?

Absolute étale cohomology?

Lichtenbaum has proposed a variant of étale cohomology, called *Weil-étale cohomology*, which would allow for the interpretation of special values. For the relationship with Deninger's formalism, see Morin's lecture.

However, there are reasons to think that étale cohomology is not the most natural way to look for a spectral interpretation of L-functions. Example (Weil): class field theory describes a reciprocity map

$$\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})/\mathbb{Q}^\times = \left({\prod_v}' \mathbb{Q}_v^\times \right) /\mathbb{Q}^\times \to \mathsf{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}),$$

but this map fails to interpret the archimedean part of the idèle class group.

Absolute de Rham cohomology?

I'm instead looking for an analogue of rigid cohomology (*absolute de Rham cohomology*) that might provide a spectral interpretation of L-functions. In such a construction, archimedean and nonarchimedean places should enter on comparable footing; for instance, *Hodge theory* and *p-adic Hodge theory* would play corresponding roles in determining Euler factors.

Various clues from arithmetic geometry point towards extracting absolute de Rham cohomology from an appropriate version of the *de Rham-Witt complex*, and suggest how one might get started doing that.

Contents

p-typical Witt vectors

Fix a prime number p. The *p-typical Witt vectors* are an endofunctor W_p on rings, such that $W_p(R)$ has underlying set $R^{\{0,1,\dots\}}$, and the *ghost map*

$$(r_0, r_1, r_2, \dots) \mapsto (r_0, r_0^p + pr_1, r_0^{p^2} + pr_1^p + p^2 r_2, \dots)$$

is a natural transformation of rings for the product ring structure on the target. For R a perfect $\mathbb{F}_p$-algebra, $W_p(R)$ is the unique strict p-ring with $W_p(R)/pW_p(R) \cong R$.

If R is of characteristic p and carries a submultiplicative norm, one obtains *overconvergent* subrings of $W_p(R)$ by imposing growth conditions of the form $|r_i|^{p^{-i}} \leq ab^i$ (thanks to homogeneity of Witt vector arithmetic). This is true in a limited form for more general R.

The big Witt vectors

The *big Witt vectors* are an endofunctor $\mathbb{W}$ on rings, such that $\mathbb{W}(R)$ has underlying set $R^{\mathbb{N}}$ (for $\mathbb{N}$ the positive integers), and the *ghost map*

$$(r_n)_{n\in\mathbb{N}} \mapsto (w_n)_{n\in\mathbb{N}}, \qquad w_n = \sum_{d|n} d r_d^{n/d}$$

is a natural transformation of rings for the product ring structure on the target. The ring $\mathbb{W}(R)$ projects onto $W_p(R)$ for each prime p, and carries special operators F_n, V_n for $n \in \mathbb{N}$ (*Frobenius* and *Verschiebung*). There is also a multiplicative map $R \to \mathbb{W}(R)$ taking r to $[r] = (r, 0, 0, \dots)$ (the *Teichmüller map*).

The rings $\mathbb{W}(R)$ are examples of λ*-rings*. In Borger's philosophy, λ-ring structures stand in for descent data from $\mathrm{Spec}(\mathbb{Z})$ to $\mathrm{Spec}(\mathbb{F}_1)$, for $\mathbb{F}_1$ the mysterious *field of one element*. This extends the idea that λ_p-ring structures provide descent data from $\mathrm{Spec}(\mathbb{Z}_p)$ to $\mathrm{Spec}(\mathbb{F}_p)$.

The big de Rham-Witt complex (after Hesselholt)

The module of Kähler differentials $\Omega_{\mathbb{W}(R)}$ is naturally a λ-module over $\mathbb{W}(R)$, so the Frobenius maps F_n act naturally. (This is not the action given by viewing F_n as a ring endomorphism!) Form the quotient of the tensor algebra of $\Omega_{\mathbb{W}(R)}$ over $\mathbb{W}(R)$ by the relations

$$da \otimes da - d\log[-1] \otimes F_2(da) \qquad (a \in R);$$

this is the exterior algebra if $2^{-1} \in R$.

There is a natural further quotient $\mathbb{W}\Omega^{\cdot}_R$ on which the maps V_n extend and satisfy

$$F_n dV_n(\omega) = d\omega + (n-1)(d\log[-1]) \cdot \omega \qquad (\omega \in \mathbb{W}\Omega^{\cdot}_R).$$

This is the *big absolute de Rham-Witt complex*.

Variants of de Rham-Witt

From Hesselholt's construction, one can recover other de Rham-Witt complexes appearing in the literature. These complexes appear in a variety of ways and for a variety of reasons.

- Inspired by Bloch's *p-typical curves in K-theory* plus ideas of Lubkin, Deligne-Illusie introduced p-typical de Rham-Witt to compute crystalline cohomology of smooth proper schemes over a field of characteristic p. It also inspired their proof of degeneration of the Hodge-de Rham spectral sequence in characteristic 0.
- The p-typical construction was modified by Davis-Langer-Zink by introducing overconvergent Witt vectors, in order to compute rigid cohomology of smooth schemes over a field of characteristic p. This might work for stacks too (Brown-Davis).
- The p-typical construction behaves well for $\mathbb{Z}_{(p)}$-algebras. This can be used, for instance, to define Frobenius and monodromy operators on p-adic étale cohomology for schemes over $\mathbb{Q}_p$ (Hyodo-Kato).

Variants of de Rham-Witt (continued)

- A *relative de Rham-Witt complex* was introduced by Langer and Zink to study relative crystalline cohomology.
- Absolute de Rham-Witt was introduced by Hesselholt and Madsen in order to compute *topological Hochschild cohomology*, with applications to algebraic K-theory via the *cyclotomic trace map*. Their definition was a bit off at the prime 2; this was fixed by Costeanu.
- One can also interpret de Rham-Witt naturally in terms of homotopical algebra. See Barwick's talk.
- There might even be links to string theory! See Stienstra's talk.

This (incomplete) list suggests the centrality of the de Rham-Witt construction.

Absolute de Rham-Witt and absolute cohomology

The differential graded algebra $\mathbb{W}\Omega^{\cdot}_{R}$ only contains $\mathbb{Z}$ in its center, not $\mathbb{W}(\mathbb{Z})$ (in contrast with the Langer-Zink construction). It thus seems to be trying to compute *crystalline cohomology over* $\mathbb{F}_1$.

A further consistency with Connes's program is that $\mathbb{W}\Omega^{\cdot}_{\mathbb{Z}}$ is a quotient of $\Omega^{\cdot}_{\mathbb{W}(\mathbb{Z})}$ by an explicit differential graded ideal generated in degree 1. In other words, de Rham-Witt defines a *foliation* on $\mathbb{W}(\mathbb{Z})$ compatible with Frobenius and Verschiebung; in Borger's philosophy, this corresponds to a foliation on Spec($\mathbb{Z}$) relative to $\mathbb{F}_1$.

To understand *L*-functions, one must integrate complex analysis into the construction. To understand how to do this, it will help to see how *nonarchimedean analytic geometry* already plays a role.

Contents

Several approaches to nonarchimedean analytic geometry

Analytic geometry can be thought of as the analogue of algebraic geometry in which the basic spaces correspond not to bare rings, but to *Banach rings* (rings complete for submultiplicative norms).

- Tate turns these rings into *maximal spectra* (spaces of maximal ideals). Raynaud's *formal geometry* is related.
- Berkovich turns them into *Gel'fand spectra* (spaces of multiplicative real-valued seminorms).
- Huber turns them into *adic spectra* (spaces of multiplicative seminorms valued in totally ordered groups). The approach of Fujiwara-Kato is related.

Since I'm interested in links with *archimedean* analytic geometry, I tend to favor Berkovich's approach. But I would welcome counterarguments!

Gel'fand spectra

Let A be a commutative Banach ring. Berkovich associates to A the compact topological space $\mathcal{M}(A)$ of multiplicative seminorms on A bounded by the given norm. (The resulting spaces are related to the polyhedral spaces used in *tropical geometry*.)

This construction is usually made over a *nonarchimedean analytic field* like $\mathbb{Q}_p$, in which case one can restrict to *affinoid algebras* and recover a theory of analytic spaces. See Temkin's lecture.

However, at some price, one can work absolutely. In fact, there is no need to restrict to nonarchimedean seminorms!

Nonarchimedean geometry of Witt vectors

Here is an example of an absolute construction in the theory of Berkovich spaces. Let R be a perfect $\mathbb{F}_p$-algebra carrying a power-multiplicative norm α, and let $\mathfrak{o}_R$ be the subring of elements of norm at most 1. Then the formula

$$(r_0, r_1, \dots) \mapsto \sup_i \{p^{-i}\alpha(r_i)^{p^{-i}}\}$$

defines a power-multiplicative norm $\lambda(\alpha)$ on $W_p(\mathfrak{o}_R)$. The map $\lambda : \mathcal{M}(\mathfrak{o}_R) \to \mathcal{M}(W_p(\mathfrak{o}_R))$ sections the projection $\mu : \mathcal{M}(W_p(\mathfrak{o}_R)) \to \mathcal{M}(\mathfrak{o}_R)$ defined by

$$\mu(\beta)(r) = \beta([r]).$$

In fact, there is a natural homotopical retraction of $\mathcal{M}(W_p(\mathfrak{o}_R))$ onto image(λ). This construction has consequences in p-adic Hodge theory.

Contents

Comparison isomorphisms

Traditional Hodge theory revolves around the relationship between Betti and de Rham cohomology on a complex algebraic variety. The subject of *p-adic Hodge theory* concerns a similar relationship between p-adic étale cohomology and de Rham cohomology for varieties over p-adic fields.

For instance, let X be a smooth proper scheme over $\mathbb{Z}_p$. For a certain topological $\mathbb{Q}_p$-algebra $\mathbf{B}_{\mathrm{crys}}$, one obtains a distinguished isomorphism

$$H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}} \cong H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{crys}}$$

from work of Fontaine, Faltings, Tsuji, Niziol, etc.

Another way to say this is that de Rham cohomology can be recovered from étale cohomology, and vice versa, using certain p-adic analytic constructions.

Galois representations and vector bundles

A key component of p-adic Hodge theory is the analytic description of continuous representations of p-adic Galois groups on finite dimensional $\mathbb{Q}_p$-vector spaces. This has commonly been done using (φ, Γ)-*modules* (Fontaine, Colmez, Berger, etc.).

It appears from work of Fargues-Fontaine (building on Kedlaya, Berger, Kisin, etc.) that one can instead use Γ-*equivariant vector bundles* on a certain one-dimensional scheme (the *curve in p-adic Hodge theory*). One uses only those which are *semistable of degree* 0, i.e., of degree 0 with no subbundles of positive slope. This evokes the Narasimhan-Seshadri correspondence between stable degree 0 vector bundles on a compact Riemann surface and irreducible unitary representations of π_1.

Geometry of the curve in p-adic Hodge theory

The study of $\mathcal{M}(W_p(R))$ from the previous section implies that the Fargues-Fontaine curve, when analytified in a natural way, has the homotopy type of a circle. Better yet, one of the generators of the fundamental group corresponds to the Frobenius map on $\mathbf{B}_{\text{crys}}$!

Wild question: is this curve the fibre at p of a scheme over $\mathrm{Spec}(\mathbb{Z})$ whose quotient by a one-dimensional group action is Connes's *arithmetic curve*? And can one try to construct the whole arithmetic curve using a big Witt vector analogue of the construction of Fontaine's period rings?

A first step would be to rewrite the p-adic period rings in such a way that all references to p pass through the p-adic absolute value. It should be possible to do this by working directly with the Witt vectors of $\mathbb{Q}_p$ and its extensions (Davis-Kedlaya).

Relative p-adic Hodge theory

One should maybe think of p-adic étale and de Rham together as a *single* cohomology theory. It would be helpful to make this theory work well in the relative setting. Work of Faltings (notably the *almost purity theorem*) provides some partial answers (Andreatta, Brinon, Iovița, etc.). An alternate approach (Kedlaya-Liu) goes through the following construction.

Let k be a perfect field of characteristic p, and equip $k((\overline{\pi}))$ with the $\overline{\pi}$-adic norm with $|\overline{\pi}| = p^{-p/(p-1)}$. Let R be any reduced affinoid algebra over $k((\overline{\pi}))$ (or a completed direct limit of same), with the spectral norm. For $z = \sum_{i=0}^{p-1} [1+\overline{\pi}]^{i/p}$, there is a natural homeomorphism

$$\mathcal{M}(R) \cong \mathcal{M}(W_p(\mathfrak{o}_R)[[\overline{\pi}]^{-1}]/(z)).$$

This can be used to show that the two rings have equivalent categories of finite étale algebras, étale $\mathbb{Z}_p$-local systems, and étale $\mathbb{Q}_p$-local systems.

Links to de Rham-Witt?

It would help greatly to understand how constructions in p-adic Hodge theory can be described in terms of de Rham-Witt. For example, for X a smooth proper $\mathbb{Z}_p$-scheme, absolute de Rham-Witt provides something like a *Gauss-Manin connection* on the relative de Rham-Witt cohomology. Can one use this connection to get back to p-adic étale cohomology, by taking horizontal sections?

Far wilder idea: can one do something like this globally to get to Weil-étale cohomology? Could this even provide a construction of the latter?

Intermediate question: can one use the language of Fontaine's rings with p replaced by the infinite place to articulate ordinary Hodge theory?

Contents

What next?

That's where you come in. Help me figure out where to go with this!

Thanks for listening. Enjoy the conference!