Towards a database of hypergeometric L-functions

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

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Hasse–Weil L-functions

Let X be a classical[1] algebraic variety over $\mathbb{Q}$ (for simplicity). For $w = 0, \dots, 2\dim(X)$, we get an associated **(incomplete) Hasse–Weil L-function** built out of **Euler factors**:

$$L_w(X,s) = \prod_p L_p(X, p^{-s})^{-1} \quad (\mathrm{Real}(s) \gg 0), \quad L_p(X,T) := \det(1 - T\,\mathrm{Frob}_p, H^w_{\mathrm{et}}(X, \mathbb{Q}_\ell)^{I_p}).$$

We can similarly define $L(M,s)$ for M a **motive**[2] factor of $H^w(X)$; we refer to w as the **weight** of M and $d = \dim M$ as the **dimension**. For instance, if X is a classical curve, then $H^1(X)$ splits according to the isogeny decomposition of the Jacobian $J(X)$.

The prime p is **good** for M if I_p acts trivially, else **bad**. We have $\deg(L_p(X,T)) \leq d$ with equality iff p is good. Using the ramification filtration on I_p, we define the **conductor** N as a certain product of powers of the bad primes.

[1] Smooth, proper, and geometrically irreducible. Sometimes called "nice".

[2] This is as much as you need to know about what a motive is for this talk! It's a long messy story.

Expected properties

There is an "Euler factor at infinity" given as a certain product of Gamma factors determined (easily) by the Hodge numbers of M. Adding these plus the conductor factor $N^{s/2}$ gives the **completed L-function** $\Lambda(M, s)$ which conjecturally admits a meromorphic continuation to $\mathbb{C}$ satisfying the functional equation

$$\Lambda(M, d+1-s) = \epsilon\Lambda(M, s), \qquad \epsilon \in \{\pm 1\}.$$

By analogy with the Riemann hypothesis, we also expect all zeroes of $\Lambda(M, s)$ to lie on the axis of symmetry $\mathrm{Real}(s) = (d+1)/2$.

It is natural to consider features of these L-functions: zero distribution, special values (as in the conjecture of Birch and Swinnerton–Dyer questions to those commonly asked about the Riemann zeta function or Dirichlet L-functions. However, this would be greatly assisted by some numerical data...

Example: elliptic curves

For X a curve of genus 1 and $w = 1$, for p good,

$$L_p(X, T) = 1 - a_p T + pT^2, \qquad a_p = p + 1 - \#X(\mathbb{F}_p).$$

The bad Euler factors and conductor exponents can be computed using **Tate's algorithm**.

The analytic continuation and functional equation for $L_1(X, s)$ is known; it follows from the **modularity of elliptic curves** (Wiles, Taylor–Wiles, et al.). This allows for rapid tabulation of elliptic curves with bounded conductor (Cremona).

The value at $s = 1$ is explained by the conjecture of Birch and Swinnerton–Dyer. This is known in many cases.

The analogue of the Riemann hypothesis is known in no cases!

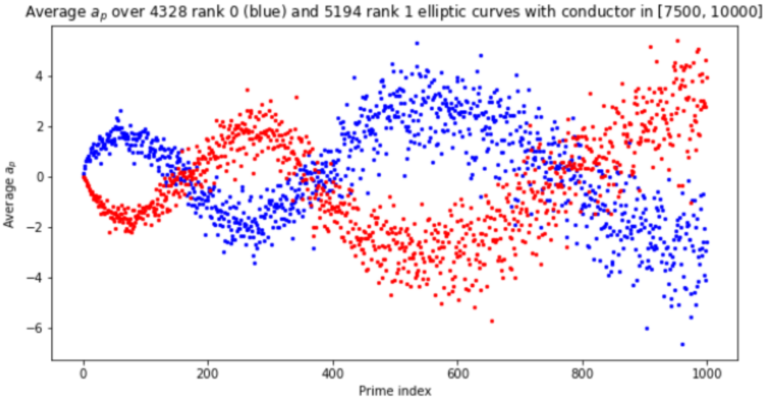
A diversity problem

We have very good technology to compute Hasse–Weil L-functions in certain cases, e.g., curves (Kyng).

However, for $w > 1$, we are practically limited to varieties whose de Rham cohomology can be managed easily (e.g., nondegenerate smooth hypersurfaces in toric varieties). This in turn limits the options for the Hodge numbers.

However, there are interesting phenomena to be explored if we can collect more diverse data...

Example: the murmurations phenomenon



This graphic is due to He–Lee–Oliver–Pozdnyakov. It features L-functions of elliptic curves; can it be replicated in other settings?

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Hypergeometric data

For $\alpha, \beta \in (\mathbb{Q} \cap [0,1))^n$ with $\alpha_i - \beta_j \notin \mathbb{Z}$ for all i, j, there is an irreducible variation of Hodge structures of rank n on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ for one of whose periods the Picard–Fuchs equation is the hypergeometric differential equation

$$P(\alpha;\beta)(z\frac{d}{dz})(y) = 0, \quad P(\alpha;\beta)(D) := z\prod_{i=1}^{n}(D+\alpha_i) - \prod_{j=1}^{n}(D+\beta_j-1).$$

The Hodge vector/motivic weight can be read from the **zigzag function**

$$Z_{\alpha,\beta}(x) := \#\{j : \alpha_j \leq x\} - \#\{j : \beta_j \leq x\}.$$

See for instance this example in LMFDB.

Hereafter we assume that α, β are **Galois-stable**,[3] meaning that the multiplicity of any $\frac{r}{s} \in \mathbb{Q}$ (in lowest terms) depends only on s. LMFDB includes all balanced HG data with $n \leq 10$.

[3] Otherwise we get motives defined only over some abelian extension of $\mathbb{Q}$.

L-functions

For α, β Galois-stable, this variation of Hodge structures arises from a family of Chow motives[4] $M^{\alpha,\beta}$ over $\mathbb{Q}$.

For any given $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, the motive $M_z^{\alpha,\beta}$ has bad reduction[5] at these primes:

- **wild** primes p, at which α or β is not in $\mathbb{Z}_{(p)}^n$;
- **tame** primes p, which are not wild but either z or $z-1$ is not a p-adic unit.

For such z, we obtain an associated L-function; our goal is to compute these L-functions **at scale** in order to exhibit them in LMFDB.

Since there are few bad primes, the only difficulties in computing bad Euler factors (and conductor exponents) are theoretical[6]. We thus focus on good primes.

[4] There are various explicit realizations; see Beukers–Cohen–Mellit, Kelly–Voight, etc. There are many special parameter sets that correspond to more familiar objects like hyperelliptic curves, K3 surfaces, Calabi–Yau threefolds, etc.

[5] This is only an upper bound; there can be a "wild" or "tame" prime at which the reduction is actually good.

[6] Precise formulas for wild primes are ongoing work of Roberts–Rodriguez Villegas.

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Trace formula

For q a power of a good p, let $H_q\left(\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\middle| z\right)$ be the trace of Frob_q on $M_z^{\alpha,\beta}$. By work of Greene, Katz, Beukers–Cohen–Mellit, Cohen–Rodriguez Villegas–Watkins, etc., we have a formula:[7]

$$H_q\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) = \frac{1}{1-q}\sum_{m=0}^{q-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}q^{D+\xi_m(\beta)}\left(\prod_{j=1}^{n}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)[z]^m, \text{ where}$$

- η_m, ξ_m, D denote some combinatorial quantities (see below);
- $(x)_m^*$ is a p-adic analogue of the Pochhammer symbol (see below);
- $[z] \in \mathbb{Q}_p^{\mathrm{unr}}$ is the multiplicative lift[8] of z.

For fixed q, this is all easy to compute (implemented in Magma and SageMath).

[7] The original formula of this form is based on finite hypergeometric sums, which contain Gauss sums. The contribution of CRVW is to reformulate using the Gross–Koblitz formula.

[8] Proposed replacement terminology for the historical term "Teichmüller lift".

Combinatorial quantities in the trace formula

$$H_q\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-q}\sum_{m=0}^{q-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}q^{D+\xi_m(\beta)}\left(\prod_{j=1}^{n}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)[z]^m$$

The powers of $-p$ and $q = p^f$ are expressed in terms of the following:[9]

$$\eta_m(x_1,\dots,x_n) := \sum_{j=1}^{n}\sum_{v=0}^{f-1}\left\{p^v\left(x_j+\tfrac{m}{1-q}\right)\right\}-\{p^v x_j\},\ \ \{x\} := x-\lfloor x\rfloor;$$

$$\xi_m(\beta) := \#\{j:\beta_j=0\}-\#\left\{j:\beta_j+\tfrac{m}{1-q}=0\right\};$$

$$D := \frac{w+1-\#\{j:\beta_j=0\}}{2}.$$

In particular, if we break up $[0,1)$ at the values in $\alpha\cup\beta$, then the powers of $-p$ and q remain constant as $\frac{m}{q-1}$ varies within a subinterval.

[9] This assumes $0\notin\alpha$. Otherwise, swap $\alpha\leftrightarrow\beta$ and $z\leftrightarrow 1-z$.

Pochhammer symbols in the trace formula

In the formula

$$H_q\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right) = \frac{1}{1-q}\sum_{m=0}^{q-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}q^{D+\xi_m(\beta)}\left(\prod_{j=1}^{n}\frac{(\alpha_j)^*_m}{(\beta_j)^*_m}\right)[z]^m$$

the analogue of the Pochhammer symbol is given by

$$(x)^*_m := \frac{\Gamma^*_q\left(x+\frac{m}{1-q}\right)}{\Gamma^*_q(x)}, \qquad \Gamma^*_q(x) := \prod_{v=0}^{f-1}\Gamma_p(\{p^v x\})$$

where $\Gamma_p\colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is the Morita p-adic Gamma function. In particular, Γ_p is continuous, $\Gamma_p(0) = 1$, and

$$\Gamma_p(x+1) = \begin{cases} -x\Gamma_p(x) & x \notin p\mathbb{Z}_p \\ -\Gamma_p(x) & x \in p\mathbb{Z}_p. \end{cases}$$

The prime case

Let us now focus on the case $q = p$. In the formula

$$H_p\left(\begin{matrix}\alpha\\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-p}\sum_{m=0}^{p-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}p^{D+\xi_m(\beta)}\left(\prod_{j=1}^{n}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)[z]^m,$$

if we restrict to summands where $\frac{m}{p-1}$ lies between two consecutive values in $\alpha \cup \beta$, then this looks like a truncated hypergeometric series.

Remember that we need to compute this for all good $p \leq X$. If we did this individually, each sum would be over $p-1$ terms, so this would cost roughly $O(X^2)$ time; however, there is clearly a great deal of redundancy. Our goal will be to leverage this redundancy to get this down to $O(X^{1+\epsilon})$.

Note that this still leaves $O(X^{3/2})$ work to deal with higher powers. It may be possible to use a similar approach to reduce this exponent also.

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A minimal example: Wilson primes

The Alhazen–Wilson theorem says that for every prime p, $(p-1)! \equiv -1 \pmod{p}$. A **Wilson prime** is a prime for which $(p-1)! \equiv -1 \pmod{p^2}$. The only known examples are $p = 5, 13, 563$.

Costa–Gerbicz–Harvey computed the reduction of $(p-1)! + 1 \bmod p^2$ for all $p \leq X$ with $X = 2 \times 10^{13}$, using a novel technique to reduce the complexity from $O(X^{2+\epsilon})$ to $O(X^{1+\epsilon})$. Harvey–Sutherland described this in terms of **accumulating remainder trees**, loosely inspired by the structure of the **fast Fourier transform** (FFT) algorithm.

To a first approximation, the idea is to replace the separate computation of $(p-1)! + 1 \pmod{p^2}$ with the serial computation of

$$n! \pmod{\prod_{n<p\leq X} p^2} \qquad \text{for } n = 0, \dots, X-1$$

to eliminate redudancy. However, this must be balanced against making the moduli so large that they slow down the computation.

Accumulating remainder trees

Say we are given integers (or matrices) $A_0, \dots, A_{b-1}$ and integers $m_1, \dots, m_{b-1}$, and we want to compute simultaneously

$$C_j := A_0 \cdots A_{j-1} \pmod{m_j} \qquad (j = 0, \dots, b-1).$$

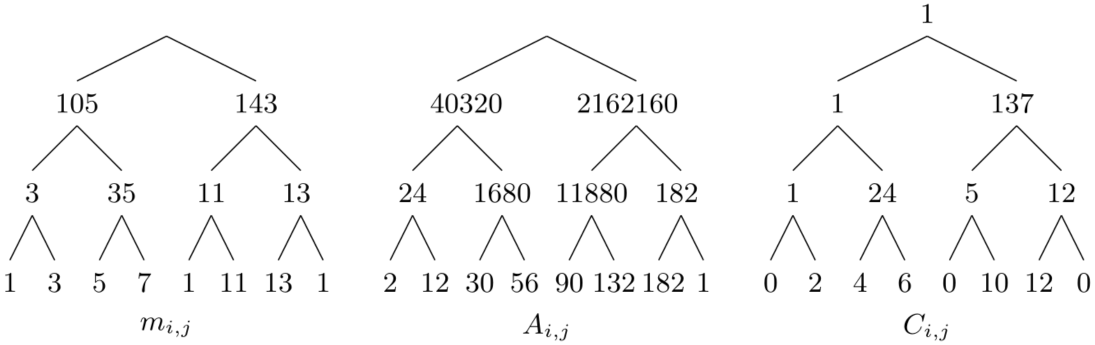
To simplify, assume $b = 2^\ell$. Form a complete binary tree of depth ℓ with nodes (i, j) where $i = 0, \dots, \ell$ and $j = 0, \dots, 2^{i-1}$. By computing from the leaves to the root, we can compute products over dyadic ranges:

$$m_{i,j} := m_{j2^{\ell-i}} \cdots m_{(j+1)2^{\ell-i}-1},$$
$$A_{i,j} := A_{j2^{\ell-i}} \cdots A_{(j+1)2^{\ell-i}-1}.$$

Then from the root to the leaves, we compute the products $C_{i,j} := A_{i,0} \cdots A_{i,j-1} \pmod{m_{i,j}}$ by writing

$$C_{i,j} = \begin{cases} C_{i-1,\lfloor j/2 \rfloor} \pmod{m_{i,j}} & j \equiv 0 \pmod 2 \\ C_{i-1,\lfloor j/2 \rfloor} A_{i,j-1} \pmod{m_{i,j}} & j \equiv 1 \pmod 2. \end{cases}$$

Illustration (Harvey–Sutherland, 2014)



Example: harmonic sums

By forming a product of the matrices $\begin{pmatrix} i^j & 0 \\ 1 & i^j \end{pmatrix}$, for any $\gamma \in \mathbb{Q} \cap (0,1]$ and e, we can efficiently compute for all $p \leq X$ the sums

$$H_{j,\gamma}(p) = \sum_{i=1}^{\lceil \gamma p \rceil - 1} i^{-j} \pmod{p^e} = \sum_{i=1}^{\lceil \gamma p \rceil - 1} \frac{(i!)^j}{((i+1)!)^j} \pmod{p^e}.$$

By applying the functional equation to obtain

$$\log \frac{\Gamma_p(x + \lceil \gamma p \rceil)}{\Gamma_p(\lceil \gamma p \rceil)} = \log \Gamma_p(x) - \sum_{j=1}^{\infty} \frac{(-x)^j}{j} H_{i,\gamma}(j),$$

for any fixed γ we can efficiently compute series expansions of Γ_p around γ modulo p^e for all $p \leq X$.

Applications in p-adic cohomology

Harvey first observed that the remainder tree technique could be used to speed up computation of L-functions via p-adic cohomology, by exploiting similar redundancies. Further work in this direction has been done by Harvey–Sutherland.

Our application to hypergeometric L-functions is more in the spirit of Costa–Gerbicz–Harvey: we amortize the computation of the trace formula modulo p^e for all $p \leq X$ by exploiting the similarity to a truncated hypergeometric sum. For $e = 1$, this will look very similar to the algorithm for harmonic sums.

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Breaking the trace formula into ranges

Returning to the hypergeometric trace formula with $q = p$:

$$H_p\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right) = \frac{1}{1-p}\sum_{m=0}^{p-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}p^{D+\xi_m(\beta)}\left(\prod_{j=1}^{n}\frac{(\alpha_j)^*_m}{(\beta_j)^*_m}\right)[z]^m,$$

Label the elements of $\alpha \cup \beta \cup \{0,1\}$ as $0 = \gamma_0 < \cdots < \gamma_s = 1$; set $m_i := \lfloor \gamma_i(p-1) \rfloor$; and focus on the sum over $m \in [m_i, m_{i+1})$ for some i. As noted earlier, there are integers σ_i, τ_i such that

$$(-p)^{\eta_m(\alpha)-\eta_m(\beta)}p^{D+\xi_m(\beta)} = \begin{cases}\tau_i & m = m_i \\ \sigma_i & m_i < m < m_{i+1}.\end{cases}$$

We can thus fix i and focus on computing, for all $p \leq X$,

$$\sum_{m=m_i+1}^{m_{i+1}-1}\left(\prod_{j=1}^{n}\frac{(\alpha_j)^*_m}{(\beta_j)^*_m}\right)[z]^m.$$

Change of endpoints

We need to shift indices so that the sums all run from 1. That is, we want to take $m = m_i + k$ and sum over $k = 1, \dots, m_{i+1} - m_i - 1$.

Write $\gamma_i = \frac{a_i}{b_i}$ in lowest terms, fix $c \in (\mathbb{Z}/b_i\mathbb{Z})^\times$, and restrict attention to $p \equiv c \pmod{b_i}$. We then have

$$m_i = \gamma_i(p-1) - \gamma_{i,c} \text{ where } a_i(p-1) = m_i b_i + r_i, \gamma_{i,c} = \frac{r_i}{b_i} \in [0,1).$$

For $\gamma \in \alpha \cup \beta$, $(\gamma)^*_m = \Gamma_p(\{\gamma + \frac{m}{1-p}\})/\Gamma_p(\gamma)$ and

$$\left\{\gamma + \frac{m}{1-p}\right\} = k + (k - \gamma_{i,c})\tfrac{p}{1-p} + h_c(\gamma, \gamma_i)$$

where

$$h_c(\gamma,\gamma_i) := \gamma - \gamma_i + \iota(\gamma,\gamma_i) - \gamma_{i,c} \in (-1,1], \quad \iota(x,y) := \begin{cases} 1 & x \le y \\ 0 & x > y. \end{cases}$$

The situation mod p

Recall that we need to sum for all $p \leq X$,

$$\sum_{m=m_i+1}^{m_{i+1}-1} \left(\prod_{j=1}^{n} \frac{(\alpha_j)^*_m}{(\beta_j)^*_m} \right) [z]^m.$$

Say we only want the trace modulo p for each $p \leq X$. Then we are reduced to summing

$$\sum_{k=1}^{m_{i+1}-m_i-1} \prod_{j=0}^{k-1} \frac{z_f f_{i,c}(j)}{z_g g_{i,c}(j)} \pmod{p},$$

where $z = \frac{z_f}{z_g}$ in lowest terms and for some positive integer b,

$$f_{i,c}(k) := b \prod_{j=1}^{n} (h_c(\alpha_j, \gamma_i) + k), \qquad g_{i,c}(k) := b \prod_{j=1}^{n} (h_c(\beta_j, \gamma_i) + k).$$

The situation mod p (continued)

Using a remainder tree, we can compute products of matrices of the form

$$A_{i,c}(k) := \begin{pmatrix} z_g g_{i,c}(k) & 0 \\ z_g g_{i,c}(k) & z_f f_{i,c}(k) \end{pmatrix}.$$

For

$$S_i(p) := A_{i,c}(1) \cdots A_{i,c}(m_{i+1} - m_i - 1),$$

we have

$$\frac{S_i(p)_{21}}{S_i(p)_{11}} \equiv \sum_{k=1}^{m_{i+1}-m_i-1} \prod_{j=0}^{k-1} \frac{z_f f_{i,c}(k)}{z_g g_{i,c}(k)} \equiv \sum_{m=m_i+1}^{m_{i+1}-1} \left(\prod_{j=1}^{n} \frac{(\alpha_j)^*_m}{(\beta_j)^*_m} \right) [z]^m \pmod{p}.$$

This is **extremely** fast in practice (see our paper from ANTS XIV, 2020).

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Some complications

In the general case, it is sufficient to compute modulo p^e for $e = \lfloor (w+1)/2 \rfloor$ where w is the motivic weight (at least for $p > 4n^2$). There are several additional complications to be overcome.

- We cannot ignore the difference between $[z]$ and z. It is easy to compute $[z]$ for any given p, but it does not behave uniformly.
- We need to incorporate the expansion of Γ_p around some rational arguments (which we already know how to compute in average polynomial time).
- The functional equation relates $\Gamma_p(x)$ to $\Gamma_p(x+1)$, not $\Gamma_p(x + \frac{1}{1-p})$.

The solution we describe here was presented at ANTS XVI in July 2024.

Harvey's generic prime construction

A key idea comes from the work of Harvey: consider products of matrices over $\mathbb{Z}[x]/(x^e)$ instead of $\mathbb{Z}$. Then for each prime p, we can take the result and replace x with something divisible by p which does **not** need to be computed by a matrix product.

For example, if the only issue were the discrepancy between z and $[z]$, we could replace $[z]$ with $z(1+x)$ and then afterwards substitute $x \mapsto [z]/z - 1$, which we can compute efficiently for individual p. (In Harvey's setting he needs to substitute $x \mapsto p$.)

In practice, we instead replace $\mathbb{Z}$ with the **noncommutative** ring of lower triangular $e \times e$ matrices over $\mathbb{Z}$. This contains $\mathbb{Z}[x]/(x^e)$ (as banded matrices) but allows for additional operations, crucially including $x \mapsto cx$.

Factorization of the quotient

The ratio of the k-th term in our sum to the 1st term can be interpreted as

$$[z]^{k-1} \prod_{\gamma \in \beta}^{\gamma \in \alpha} \frac{\Gamma_p\left(h_c(\gamma, \gamma_i) + k + \frac{(k-\gamma_{i,c})p}{1-p}\right)}{\Gamma_p\left(h_c(\gamma, \gamma_i) + 1 + \frac{(1-\gamma_{i,c})p}{1-p}\right)}$$

where $\prod_{\gamma\in\beta}^{\gamma\in\alpha}$ means take the product over $\gamma = \alpha_1, \dots, \alpha_n$ divided by the product over $\gamma = \beta_1, \dots, \beta_n$. In terms of the power series

$$R_i(x) := \prod_{\gamma \in \beta}^{\gamma \in \alpha} \frac{\Gamma_p(x + h_c(\gamma, \gamma_i) + 1)}{\Gamma_p(h_c(\gamma, \gamma_i) + 1)},$$

We can write the above ratio as

$$\left(\frac{[z]}{z}\right)^{k-1} \frac{R_i((k - \gamma_{i,c})\frac{p}{1-p})}{R_i((1 - \gamma_{i,c})\frac{p}{1-p})} \cdot \prod_{j=1}^{k-1} \frac{f_{i,c}(x+j)}{g_{i,c}(x+j)}\Bigg|_{x=(k-\gamma_{i,c})\frac{p}{1-p}}.$$

Factorization of the quotient (continued)

In the previous expression, the factor not involving j, namely

$$\left(\frac{[z]}{z}\right)^{k-1} \frac{R_i((k-\gamma_{i,c})\frac{p}{1-p})}{R_i((1-\gamma_{i,c})\frac{p}{1-p})},$$

depends on k in a usefully simple way: it can be written as

$$\sum_{h=0}^{e-1} c_{i,h}(p)\left((k-\gamma_{i,c})\frac{p}{1-p}\right)^h \pmod{p^e}$$

for some $c_{i,h}(p)$ independent of k. Conveniently, we do **not** have to worry about how these are computed when forming the matrix product!

Form of the matrix product

We apply remainder trees to multiply block matrices with $e \times e$ blocks:

$$A_{i,c}(k) := (\text{scalar}) \begin{pmatrix} \delta_{h_1,h_2} & 0 \\ (k-\gamma_{i,c})^{e-h_2}\delta_{h_1,h_2} & \left(\frac{f_{i,c}(x+k)}{g_{i,c}(x+k)}\right)^{[h_1-h_2]} \end{pmatrix}$$

where $f(x)^{[h]}$ means the coefficient of x^h in $f(x)$. The effect of adding $A_{i,c}(k)$ to the product is to increment (lower left)/(upper left) by

$$Q_{h_1,h_2}(k) = (k-\gamma_{i,c})^{h_2} \left(\prod_{j=1}^{k-1} \frac{f_{i,c}(x+j)}{g_{i,c}(x+j)} \right)^{[h_2-h_1]}$$

which we combine with the $c_{i,h}(p)$ to get what we want:

$$\sum_k \sum_{h_1,h_2} c_{i,e-h_1} Q_{h_1,h_2}(k) \left(\frac{p}{1-p}\right)^{e-h_2}.$$