Abelian varieties over $\mathbb{F}_2$ of prescribed order

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work in progress (draft available)

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These slides can be downloaded from https://kskedlaya.org/slides/.

Explicit Methods in Number Theory
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The UC San Diego campus sits on unceded ancestral land of the Kumeyaay Nation, whom I thank for their stewardship.

The order of an abelian variety

Let A be an abelian variety over $\mathbb{F}_q$. Then the **order of** A is given by

$$\#A(\mathbb{F}_q) = P(1)$$

where $P \in \mathbb{Z}[T]$ is the charpoly of Frobenius on A.*

By a theorem of Weil, for $g = \dim(A)$, in $\mathbb{C}[T]$ we have

$$P(T) = (T - \alpha_1) \cdots (T - \alpha_{2g})$$

where $|\alpha_i| = \sqrt{q}$ and $\alpha_{g+i} = \overline{\alpha}_i$.

Conversely, by Honda–Tate, given a polynomial $P \in \mathbb{Z}[T]$ of this form, some power of it occurs as the charpoly of Frobenius of some A. When $q = p$, P itself always occurs.

*Or more precisely, on the ℓ-adic Tate module of A for any prime $\ell \nmid q$.

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A closer look at the Weil bound

Weil's theorem implies that for $g = \dim(A)$,

$$(\sqrt{q}-1)^{2g} \leq \#A(\mathbb{F}_q) \leq (\sqrt{q}+1)^{2g}.$$

For fixed q, as $g \to \infty$ these intervals start to overlap, so there does not appear to be an obstruction to realizing **every** sufficiently large integer as the order of an abelian variety over $\mathbb{F}_q$ (of arbitrary order).

One can turn this intuition into a theorem by giving systematic constructions of Weil polynomials. This leads to results as on the next slide.

Acknowledgment: these are partly inspired by the tables of isogeny classes of abelian varieties in LMFDB (Dupuy–K–Roe–Vincent).

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Realization of orders

Theorem (Howe–K, March 2021)

Every positive integer (with no exceptions!) is the order of an abelian variety over $\mathbb{F}_2$, which may even taken to be ordinary.

Theorem (van Bommel–Costa–Li–Poonen–Smith, June 2021)

For any given q, every sufficiently large positive integer[a] is the order of an abelian variety over $\mathbb{F}_q$, which may even taken to be ordinary, geometrically simple, and principally polarizable.

[a] Of course the cutoff for "sufficiently large" depends on q as well as on which side conditions you add. In any case it is principle effective; with no side conditions you can realize all orders beyond $q^{3\sqrt{q}\log q}$.

An improved Weil bound

It is possible to improve the Weil bounds for **simple** abelian varieties. For example, Kadets (following Aubry–Haloui–Lachaud) showed that for $q > 2$, if A is simple of dimension g, then with finitely many exceptions†

$$\#A(\mathbb{F}_q) \geq 1.359^g.$$

In particular, for $q > 2$ any given positive integer can only occur as the order of **finitely many** simple abelian varieties over $\mathbb{F}_q$.

By contrast, Madan–Pal (1970s) found‡ **infinitely many** simple abelian varieties over $\mathbb{F}_2$ of order 1 (and even classified the Weil polynomials).

Kadets (2020) asked whether there are infinitely many simple abelian varieties over $\mathbb{F}_2$ of order 2. It is also natural to consider higher orders...

† Sample exceptions: over $\mathbb{F}_3$ and $\mathbb{F}_4$ there are elliptic curves of order 1.

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The main result

Theorem (K, July 2021)

For every positive integer m, there exist infinitely many simple abelian varieties over $\mathbb{F}_2$ of order m.

The method of proof is constructive: for **every** m we exhibit an explicit sequence of Weil polynomials corresponding to abelian varieties over $\mathbb{F}_2$ of order m. With some care, we can also ensure that these polynomials are (nearly) irreducible.

By contrast, I expect there are only finitely many Jacobians over $\mathbb{F}_2$ of order m, but this only seems to be known for $m = 1$ (Madan–Queen, Stirpe, Mercuri–Stirpe, Shen–Shi).

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Reduction steps

Any Weil polynomial of degree $2g$ for $q = 2$ has the form

$$T^g P(T + 2/T)$$

where $P(T) \in \mathbb{Z}[T]$ is a monic polynomial with all roots in $[-2\sqrt{2}, 2\sqrt{2}]$ and satisfies $\#A(\mathbb{F}_2) = P(3)$.

Following Madan–Pal (and R. Robinson), consider the polynomial

$$Q(T) = (-1)^{\deg P} P(3 - T),$$

which has roots in $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$ and satisfies $\#A(\mathbb{F}_2) = |Q(0)|$. The roots of $Q(x)$ are totally positive algebraic integers of small norm, with all conjugates in a short interval.

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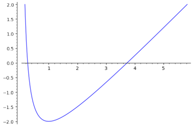
Chebyshev polynomials and a substitution

Note that $x \mapsto x + x^{-1} - 4$ carries $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$ onto $[-2, 2]$.



Let T_n be the n-th Chebyshev polynomial for the normalization

$$T_n(2\cos\theta) = 2\cos n\theta.$$

Then

$$f_n(x) := x^n T_n(x + x^{-1} - 4)$$

is a polynomial with constant term 1 and all roots in $[3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$. Madan–Pal show§ that this accounts for all AVs over $\mathbb{F}_2$ of order 1.

§By reducing to Kronecker's theorem: every algebraic integer whose complex conjugates all have norm 1 is a root of unity.

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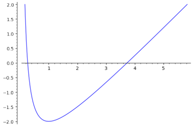

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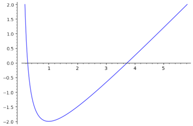

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A modified construction

Define

$$g_{n,k}(x) := (x-1)^{-k} \sum_{j=0}^{k} \binom{k}{j} f_{n+j}(x) \in \mathbb{Z}[x].$$

We will see shortly that $g_{n,k}(x)$ also has all roots in $[3-2\sqrt{2}, 3+2\sqrt{2}]$. Note that $|g_{n,k}(0)| = 2^k$.

More generally, we will give a condition on a sequence $a_0, \dots, a_k = 1$ of real numbers under which the polynomial

$$\sum_i a_i g_{n,i}(x)$$

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Sketch of a proof (via winding numbers)

Theorem (K, July 2021)

For $a_0, \dots, a_k = 1 \in \mathbb{R}$ such that $\sum_{i=0}^k a_i z^i$ all $\mathbb{C}$-roots in the disc $|z| \leq \sqrt{2}$, $P_n(x) = \sum_i a_i g_{n,i}(x)$ has all $\mathbb{C}$-roots in $[3-2\sqrt{2}, 3+2\sqrt{2}]$.

Sketch of proof: for $\theta \in [0, 2\pi]$, put $y(\theta) = e^{2\pi i \theta}$ and let $x(\theta)$ be a root of

$$x(\theta) + x(\theta)^{-1} - 4 = 2\cos\theta = y(\theta) + y(\theta)^{-1}$$

varying continuously from $3+2\sqrt{2}$ to $3-2\sqrt{2}$. Now write

$$P_n(x(\theta)) = 2x(\theta)^n \mathrm{Re}\left(y(\theta)^n s(\theta)^k \sum_{i=0}^k a_i s(\theta)^{i-k} \right), \ s(\theta) = \frac{x(\theta)y(\theta)+1}{x(\theta)-1}$$

and compute complex arguments; since $|s(\theta)| = \sqrt{2}$, the sum over i is dominated by the term $i = k$.

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A convenient choice

Each positive integer m has a unique **nonadjacent binary representation** (Reitwiesner, 1960):

$$m = \sum_{i=0}^{k} a_i 2^i \quad \text{where} \quad a_i \in \{-1, 0, 1\}, a_k = 1, \ a_i a_{i+1} = 0 \quad (i \geq 0).$$

The previous theorem applies to

$$h_{n,m}(x) := \sum_{i=0}^{k} (-1)^{i+k} a_i g_{n,i}(x),$$

for which $|h_{n,m}(0)| = m$: the nonadjacent condition implies

$$\sum_{i=0}^{k-1} |a_i| 2^{(i-k)/2} < 2^{-1} + 2^{-2} + \cdots = 1,$$

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Proof of the theorem: even order case

For any fixed choice of the a_i, the polynomials $P_n(x) = \sum_i a_i g_{n,i}(x)$ satisfy a second-order linear recurrence. This implies that any irreducible factor shared by two of the $P_n(x)$ must be a factor of some $f_n(x)$ (and so corresponds to a simple AV of order 1).

For m even, we can arrange (using either $h_{n,m}(x)$ or a slight variant) that the 2-adic Newton polygon forces an irreducible factor over $\mathbb{Q}_2$ of bounded codegree, and hence likewise over $\mathbb{Q}$. The cofactor is limited to a finite set, in which only polynomials with constant term ± 1 occur more than once; so the big irreducible factor usually has constant term $\pm m$.

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For m odd, we can force $P_n(x+1)$ to be Eisenstein at 2!

Lemma

There exists a monic integer polynomial $Q(z)$ such that:

- $Q(2) = m$;
- $Q(z) \equiv (z-1)^{\deg Q(z)} \pmod 2$; *and*
- $Q(z)$ *has all complex roots in the disc* $|z| < \sqrt{2}$.

(Then write $\sum_{i=0}^{k} a_i z^i = Q(z)$ and use these to form $P_n(x)$.)

Our proof of this is computational: we find explicit examples for $m \leq 350$, then compute larger examples by keeping track of the "quality"

$$\min\{|Q(z)| : |z| \geq \sqrt{2}\}.$$

Given enough examples of quality at least 7, we can continue via the rule

$$m \mapsto 15m + c \qquad (|c| \leq 7), \qquad Q(z) \mapsto (z^4 - 1)Q(z) + c.$$

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