The étale theta function of S. Mochizuki

Kiran S. Kedlaya

Department of Mathematics, University of California, San Diego
kedlaya@ucsd.edu
http://kskedlaya.org/slides/

The IUT theory of Shinichi Mochizuki
Clay Mathematics Institute
Oxford University, December 9-10, 2015

Reference: [EtTh] S. Mochizuki, The étale theta function and its frobenioid-theoretic manifestations, *Publ. RIMS* **45** (2009), 227–349.

Disclaimers: Some notation differs from [EtTh]. All mistakes herein are due to the speaker's misunderstandings of [EtTh]; moreover, no true understanding of IUT should be inferred. Feedback is welcome!

Supported by NSF (grant DMS-1501214), UCSD (Warschawski chair), Guggenheim Fellowship.

Setup (after Zhang's lecture)

Let C be a compact Riemann surface of genus g. Let S be a finite subset of C. Let $f: E \to C$ be a nonisotrivial elliptic fibration over $C - S$ with stable reduction at all $s \in S$. Let $\Delta_E = \sum_s \Delta_s \cdot (s)$ be the discriminant divisor; we will prove[1]

$$\deg \Delta_E \leq 6(\#S + 2g - 1).$$

To begin, fix a point $p \in C - S$; we then have a monodromy action

$$\rho : \pi_1(C - S, p) \to \mathrm{SL}(H_1(E_p, \mathbb{Z})) \cong \mathrm{SL}_2(\mathbb{Z}).$$

The group $\pi_1(C - S, p)$ is generated by a_i, b_i for $i = 1, \ldots, g$ plus a loop c_s around each $s \in S$, subject to the single relation

$$[a_1, b_1] \cdots [a_g, b_g] \prod_{s \in S} c_s = 1.$$

[1] The correct upper bound is $6(\#S + 2g - 2)$, but this requires some extra effort.

Lifting in fundamental groups

Since $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{R}/2\pi\mathbb{Z}$ via rotation, we obtain central extensions

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}) \to 1,$$
$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) \to 1.$$

The former defines an element of $H^2(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$, which we restrict along ρ: lift $\rho(a_i), \rho(b_i), \rho(c_s)$ to $\alpha_i, \beta_i, \gamma_s \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$, so that

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \prod_{s \in S} \gamma_s = m \qquad \text{for some } m \in \mathbb{Z}.$$

Measurement in fundamental groups: part 1

For $\alpha \in \widetilde{\mathrm{SL}}_2(\mathbb{R})$, define its *length* as

$$\ell(\alpha) = \sup_{x \in \mathbb{R}} |\alpha(x) - x|.$$

By taking $\alpha_i, \beta_i, \gamma_s$ to be "minimal" lifts of $\rho(a_i), \rho(b_i), \rho(c_s)$, we may ensure that

$$\ell([\alpha_i, \beta_i]) \leq 2\pi, \qquad \ell(\gamma_s) < \pi.$$

Taking lengths of both sides of the equality

$$[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \prod_{s \in S} \gamma_s = m$$

yields

$$2\pi g + \pi \# S > 2\pi |m| \Longrightarrow 2|m| \leq 2g + \# S - 1.$$

Measurement in fundamental groups: part 2

The group $\mathrm{SL}_2(\mathbb{Z})$ is generated by these two elements:

$$u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

These may be lifted to elements $\tilde{u}, \tilde{v} \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ which generate freely modulo the *braid relation* $\tilde{u}\tilde{v}\tilde{u} = \tilde{v}\tilde{u}\tilde{v}$. We thus have a homomorphism $\deg : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathbb{Z}$ taking $\tilde{u}, \tilde{v}$ to 1.

Write the discriminant divisor as $\Delta_E = \sum_{s \in S} \Delta_s \cdot (s)$. then $\rho(c_s) \sim u^{\Delta_s}$. Because of the minimal lifting, we have $\gamma_s \sim \tilde{u}^{\Delta_s}$. Since $\mathbb{Z} \subset \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ is generated by $(\tilde{u}\tilde{v})^6$, we have

$$\deg \Delta_E = \deg(m) = 12|m|.$$

Comparing with the previous slide yields the claimed inequality.

The role of theta

This argument can be viewed as a prototype for IUT, but it is better to modify it first.

We are implicitly using the interpretation $E_p \cong \mathbb{C}/(\mathbb{Z} + \tau_p\mathbb{Z})$. By exponentiating, we get $E_p \cong \mathbb{C}^\times/q_p^{\mathbb{Z}}$ for $q_p = e^{2\pi i \tau_p}$.

This isomorphism can be described in terms of ~~the~~ a Jacobi theta function

$$\vartheta_p(z) := \sum_{n\in\mathbb{Z}} (-1)^n q_p^{(n+1/2)^2} z^{2n+1}$$

via the formula

$$\wp_p(z) = (-\log \vartheta_p(z))'' + c.$$

In particular, this provides access to the symplectic structure in the guise of the Weil pairing.

The role of nonarchimedean theta

The previous discussion admits a nonarchimedean analogue for curves with split multiplicative reduction via Tate uniformization (e.g., see Mumford's appendix to Faltings-Chai).

The nonarchimedean theta function can be used to construct "something" (the *tempered Frobenioid*) playing the role of γ_s, i.e., which records the contribution of a bad-reduction prime to conductor and discriminant.

We cannot hope to do this using only the profinite (local arithmetic) étale fundamental group: this only produces the discriminant exponent as an element of $\widehat{\mathbb{Z}}$, whereas we need it in $\mathbb{Z}$ in order to make archimedean estimates. (Compare the analogous issue in global class field theory.)

Fortunately, for nonarchimedean analytic spaces, there is a natural way to "partially decomplete" the profinite étale fundamental group to obtain the *tempered fundamental group* (see Lepage's lecture).

Context: Uniformization of elliptic curves

Let K be a finite extension of $\mathbb{Q}_p$ with integral subring $\mathfrak{o}_K$. Let E be the analytification of an elliptic curve over K with split multiplicative reduction; it admits a Tate uniformization

$$E \cong \mathbb{G}_m/q^{\mathbb{Z}} \qquad \text{for some } q \in K \text{ with } v_K(q) = v_K(\Delta_{E/K}) > 0.$$

We will describe a sequence of objects over E which admit analogues on the side of formal schemes (by comparison of fundamental groups); we denote this imitation by passing from *ITALIC* (or sometimes $\mathcal{MATHCAL}$) to $\mathfrak{FRAKTUR}$ lettering. The reverse passage is the Raynaud generic fiber construction.

For example, for X as on the next slide, let $\mathfrak{X}$ be the stable model of X over $\mathfrak{o}_K$ with log structure along the special fiber.

Fundamental groups: setup

Let $\widehat{\pi}, \pi_1^{\text{top}}, \pi_1^{\text{tm}}$ denote[2] the profinite, topological, and tempered fundamental groups of an analytic space (for some basepoint).

Recall the exact sequence (on which G_K acts):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\pi}_1(\mathbb{G}_{m,\overline{K}}) & \longrightarrow & \pi_1^{\text{tm}}(E_{\overline{K}}) & \longrightarrow & \pi_1^{\text{top}}(E_{\overline{K}}) & \longrightarrow & 1 \\ & & \| & & \| & & \| & & \\ 1 & \longrightarrow & \widehat{\mathbb{Z}}(1) & \longrightarrow & \pi_1^{\text{tm}}(E_{\overline{K}}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 1. \end{array}$$

Let X be the hyperbolic log-curve obtained from E by adding logarithmic structure at the origin (the "cusp"). We have another exact sequence

$$1 \longrightarrow \pi_1^{\text{tm}}(X_{\overline{K}}) \longrightarrow \pi_1^{\text{tm}}(X) \longrightarrow G_K \longrightarrow 1.$$

Put $\pi_1^{\text{tm}}(X_{\overline{K}})^{\text{ell}} := \pi_1^{\text{tm}}(X_{\overline{K}})^{\text{ab}} := \pi_1^{\text{tm}}(E_{\overline{K}})$ and similarly for $\widehat{\pi}_1$.

[2] In [EtTh], the letters Π and Δ are generally used to distinguish arithmetic vs. geometric fundamental groups; hence the notation Δ_Θ on the next slide.

Fundamental groups: geometric Θ

The group $\widehat{\pi}_1(X_{\overline{K}})$ is profinite free on 2 generators. Consequently, if we write $\widehat{\pi}_1(X_{\overline{K}})^{\Theta}$ for the 2-step nilpotent quotient

$$\widehat{\pi}_1(X_{\overline{K}})^{\Theta} := \widehat{\pi}_1(X)/[\bullet, [\bullet, \bullet]],$$

we have natural exact sequences (the second pulled back from the first):

$$\begin{array}{ccccccccc}
1 \longrightarrow & \Delta_{\Theta} \cong \widehat{\mathbb{Z}}(1) & \longrightarrow & \pi_1^{\mathrm{tm}}(X_{\overline{K}})^{\Theta} & \longrightarrow & \pi_1^{\mathrm{tm}}(X_{\overline{K}})^{\mathrm{ell}} & \longrightarrow 1 \\
 & \| & & \downarrow & & \downarrow & \\
1 \longrightarrow & \wedge^2 \widehat{\pi}_1(X_{\overline{K}})^{\mathrm{ab}} & \longrightarrow & \widehat{\pi}_1(X_{\overline{K}})^{\Theta} & \longrightarrow & \widehat{\pi}_1(X_{\overline{K}})^{\mathrm{ell}} & \longrightarrow 1
\end{array}$$

In fact, we can write

$$\widehat{\pi}_1(X_{\overline{K}})^{\Theta} \cong \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \widehat{\mathbb{Z}}(1) \\ & 1 & \widehat{\mathbb{Z}} \\ & & 1 \end{pmatrix}, \qquad \pi_1^{\mathrm{tm}}(X_{\overline{K}})^{\Theta} \cong \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \widehat{\mathbb{Z}}(1) \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}.$$

Fundamental groups: covers and arithmetic Θ

For any (possibly infinite) connected tempered cover W of X, $\pi_1^{\text{tm}}(W)$ is an open subgroup of $\pi_1^{\text{tm}}(X)$. We obtain exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_\Theta \cap \pi_1^{\text{tm}}(W_{\overline{K}}) & \longrightarrow & \pi_1^{\text{tm}}(W_{\overline{K}})^\Theta & \longrightarrow & \pi_1^{\text{tm}}(W_{\overline{K}})^{\text{ell}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_\Theta \cap \pi_1^{\text{tm}}(W_{\overline{K}}) & \longrightarrow & \widehat{\pi}_1(W_{\overline{K}})^\Theta & \longrightarrow & \widehat{\pi}_1(W_{\overline{K}})^{\text{ell}} & \longrightarrow & 1 \end{array}$$

by taking subobjects of the corresponding objects over $X_{\overline{K}}$.

Let $L \subseteq \overline{K}$ be the integral closure of K in the function field of W. Define $\pi_1^{\text{tm}}(W)^\Theta$ as the quotient of $\pi_1^{\text{tm}}(W)$ for which

$$1 \to \pi_1^{\text{tm}}(W_{\overline{K}})^\Theta \to \pi_1^{\text{tm}}(W)^\Theta \to G_L \to 1$$

is exact. Similarly, with π_1^{tm} replaced by $\widehat{\pi}_1$ and/or $\bullet^\Theta$ replaced by $\bullet^{\text{ell}}$.

The universal topological cover

Let Y be a copy of $\mathbb{G}_m$ with log structure at $q^{\mathbb{Z}}$, viewed as an infinite étale cover of X. The exact sequence

$$1 \to \Delta_\Theta \to \pi_1^{\text{tm}}(Y_{\overline{K}})^\Theta \to \pi_1^{\text{tm}}(Y_{\overline{K}})^{\text{ell}} \to 1$$

consists of *abelian*[3] *profinite* groups; that is,

$$\widehat{\pi}_1(Y_{\overline{K}})^\Theta = \pi_1^{\text{tm}}(Y_{\overline{K}})^\Theta \cong \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \widehat{\mathbb{Z}}(1) \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

[3] This is why we replaced $\bullet^{\text{ab}}$ with $\bullet^{\text{ell}}$ earlier.

Some Galois coverings

For $N \geq 1$, define $K_N := K(\zeta_N, q^{1/N}) \subseteq \overline{K}$. Pick a cusp of Y; its decomposition group determines (up to $\pi_1^{\mathrm{tm}}(Y_{\overline{K}})$-conjugation) a section $G_K \to \pi_1^{\mathrm{tm}}(Y)^{\mathrm{ell}}$.

The image of the composition

$$G_{K_N} \hookrightarrow G_K \to \pi_1^{\mathrm{tm}}(Y)^{\mathrm{ell}} \twoheadrightarrow \pi_1^{\mathrm{tm}}(Y)^{\mathrm{ell}}/N$$

is stable under $\pi_1^{\mathrm{tm}}(X)$-conjugation; we thus obtain a Galois covering $Y_N \to Y$ and an exact sequence

$$1 \to \pi_1^{\mathrm{tm}}(Y_{\overline{K}})^{\mathrm{ell}}/N \to \mathrm{Gal}(Y_N/Y) \to \mathrm{Gal}(K_N/K) \to 1.$$

Some more Galois coverings

Define the finite[4] Galois extension $J_N := K_N(a^{1/N} : a \in K_N) \subseteq \overline{K}$.

Since any two splittings of

$$1 \to \Delta_\Theta/N \cong \mathbb{Z}/N\mathbb{Z}(1) \to \pi_1^{\mathrm{tm}}(Y_N)^\Theta/N \to G_{K_N} \to 1$$

differ by a class in $H^1(G_{K_N}, \mathbb{Z}/N\mathbb{Z}(1))$, they restrict equally to G_{J_N}. Again, we thus get a Galois covering $Z_N \to Y_N$ and an exact sequence

$$1 \to \Delta_\Theta/N \to \mathrm{Gal}(Z_N/Y_N) \to \mathrm{Gal}(J_N/K_N) \to 1.$$

[4]Because K_N is a local field.

Integral models and line bundles

Recall that X, Y, Y_N, Z_N come with integral models $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Y}_N, \mathfrak{Z}_N$.

The special fiber of $\mathfrak{Y}_N$ is an open chain of $\mathbb{P}^1$'s, so $\mathrm{Pic}(\mathfrak{Y}_N) \cong \mathbb{Z}^{\mathbb{Z}}$. Let $\mathfrak{L}_N$ be the line bundle corresponding to the constant function 1 in $\mathbb{Z}^{\mathbb{Z}}$.

Choose a section $s_1 \in \Gamma(\mathfrak{Y} = \mathfrak{Y}_1, \mathfrak{L}_1)$ whose divisor is the cusps. Fix an identification of $\mathfrak{L}_1|_{\mathfrak{Y}_N}$ with $\mathfrak{L}_N^{\otimes N}$; we then get a unique action of $\mathrm{Gal}(Y_N/X)$ on $\mathfrak{L}_1|_{\mathfrak{Y}_N}$ preserving s_1.

Proposition (EtTh, Proposition 1.1)

(i) *The section $s_1|_{\mathfrak{Y}_N} \in \Gamma(\mathfrak{Y}_N, \mathfrak{L}_1|_{\mathfrak{Y}_N})$ admits an N-th root $s_N \in \Gamma(\mathfrak{Z}_N, \mathfrak{L}_N|_{\mathfrak{Z}_N})$.*

(ii) *There is a unique action of $\pi_1^{\mathrm{tm}}(X)$ on $\mathfrak{L}_N \otimes_{\mathfrak{o}_{K_N}} \mathfrak{o}_{J_N}$ (over $\mathfrak{Y}_N \times_{\mathfrak{o}_{K_N}} \mathfrak{o}_{J_N}$) compatible with the map $\mathfrak{Z}_N \to V(\mathfrak{L}_N \otimes_{\mathfrak{o}_{K_N}} \mathfrak{o}_{J_N})$ determined by s_N (where V means the geometric line bundle). Moreover, this action factors through a faithful action of $\mathrm{Gal}(Z_N/X)$.*

Orientation

Define (with N optionally omitted when $N = 1$)

$$\ddot{K}_N := K_{2N}, \qquad \ddot{J}_N = \ddot{K}_{2N}(a^{1/N} : a \in \ddot{K}_{2N}),$$
$$\ddot{Y}_N := Y_{2N} \times_{\ddot{K}_N} \ddot{J}_N, \qquad \ddot{\mathcal{L}}_N := \mathcal{L}_N|_{\ddot{Y}_N} \cong \mathcal{L}_{2N}^{\otimes 2} \otimes_{\ddot{K}_N} \ddot{J}_N, \qquad \text{etc.}$$

We now fix an isomorphism $\mathrm{Gal}(Y/X) \cong \mathbb{Z}$ and a compatible $\mathbb{Z}$-labeling of the components of the special fiber of $\mathfrak{Y}$ (and hence of $\mathfrak{Y}_N$).

Let $\mathfrak{D}_N$ be the effective Cartier divisor on $\ddot{\mathfrak{Y}}_N$ supported on the special fiber which on component j is the divisor of $q^{j^2/(2N)}$. By counting degrees, we see that there exists a section $\tau_N \in \Gamma(\ddot{\mathfrak{Y}}_N, \ddot{\mathfrak{L}}_N)$ with zero locus $\mathfrak{D}_N$.

Lemma (EtTh, Lemma 1.2)

We may choose τ_N so $\tau_{N_1}^{\otimes N_1/N_2} = \tau_{N_2}$ whenever $N_2 | N_1$. We thus get compatible (over N) actions of $\pi_1^{\mathrm{tm}}(Y)$ on $\ddot{\mathfrak{Y}}_N, V(\ddot{\mathfrak{L}}_N)$ preserving τ_N.

The étale theta class, part 1: basic properties

The action of $\pi_1^{\mathrm{tm}}(Y)$ on $V(\ddot{\mathfrak{L}}_N)$ from [EtTh, Lemma 1.2] is not the one induced from [EtTh, Proposition 1.1]; they differ by a $2N$-th root of unity. (If we restrict to $\pi_1^{\mathrm{tm}}(\ddot{Y})$, we only get N-th roots of unity.)

Proposition (EtTh, Proposition 1.3)

Comparing the two actions, we obtain

$$\eta_N^{\Theta} \in H^1(\pi_1^{\mathrm{tm}}(Y), \Delta_{\Theta} \otimes \frac{1}{2}\mathbb{Z}/N\mathbb{Z} \cong \frac{1}{2}\mathbb{Z}/N\mathbb{Z}(1)).$$

This arises from $\pi_1^{\mathrm{tm}}(Y)/\pi_1^{\mathrm{tm}}(\ddot{Z}_N)$; the further restriction to

$$(H^1 = \mathrm{Hom})(\pi_1^{\mathrm{tm}}(\ddot{Y})/\pi_1^{\mathrm{tm}}(\ddot{Z}_N), \Delta_{\Theta} \otimes \frac{1}{2}\mathbb{Z}/N\mathbb{Z})$$

is the composite of the natural isom $\pi_1^{\mathrm{tm}}(\ddot{Y})/\pi_1^{\mathrm{tm}}(\ddot{Z}_N) \cong \Delta_{\Theta} \otimes \mathbb{Z}/N\mathbb{Z}$ with the embedding $\mathbb{Z}/N\mathbb{Z} \to \frac{1}{2}\mathbb{Z}/N\mathbb{Z}$.

The étale theta class, part 2: ambiguities

Put $\mathfrak{o}^{\times}_{K/\ddot{K}} := \{a \in \mathfrak{o}^{\times}_{\ddot{K}} : a^2 \in K\}$; this has a Kummer map to $H^1(G_K, \frac{1}{2}\mathbb{Z}/N\mathbb{Z}(1))$ extending the usual $K^{\times} \to H^1(G_K, \mathbb{Z}/N\mathbb{Z}(1))$.

Proposition (EtTh, Proposition 1.3 continued)

The set of cohomology classes

$$\mathfrak{o}^{\times}_{K/\ddot{K}} \cdot \eta^{\Theta}_N \in H^1(\pi_1^{\text{tm}}(Y), \Delta_\Theta \otimes \frac{1}{2}\mathbb{Z}/N\mathbb{Z})$$

does not depend on the choices of s_1, s_N, τ_N. In particular, these sets are compatible with changing N, so we get sets

$$\mathfrak{o}^{\times}_{K/\ddot{K}} \cdot \eta^{\Theta} \in H^1(\pi_1^{\text{tm}}(Y), \frac{1}{2}\Delta_\Theta \cong \frac{1}{2}\widehat{\mathbb{Z}}(1))$$

each arising from $\pi_1^{\text{tm}}(Y)^{\Theta}$ and restricting in $(H^1 = \text{Hom})(\widehat{\pi}_1(Y)^{\Theta}, \frac{1}{2}\Delta_\Theta)$ to the composition $\widehat{\pi}_1(Y)^{\Theta} \cong \Delta_\Theta \hookrightarrow \frac{1}{2}\Delta_\Theta$.

The étale theta class, part 3: ±-descent

Proposition (EtTh, Proposition 1.3 continued)

The restricted classes

$$\mathfrak{o}_{K/\ddot{K}}^{\times} \cdot \eta^{\Theta}|_{\ddot{Y}} \in H^1(\pi_1^{\mathrm{tm}}(\ddot{Y}), \frac{1}{2}\Delta_{\Theta})$$

are "integral", i.e., they arise from classes

$$\mathfrak{o}_{\ddot{K}}^{\times} \cdot \ddot{\eta}^{\Theta} \in H^1(\pi_1^{\mathrm{tm}}(\ddot{Y}), \Delta_{\Theta} = \widehat{\mathbb{Z}}(1)).$$

Any element of any of the sets $\mathfrak{o}_{K/\ddot{K}}^{\times} \cdot \eta_N^{\Theta}, \mathfrak{o}_{K/\ddot{K}}^{\times} \cdot \eta^{\Theta}, \mathfrak{o}_{\ddot{K}}^{\times} \cdot \ddot{\eta}^{\Theta}$ is called "the" *étale theta class*.

Note: the $\frac{1}{2}$ is forced because the divisor $\mathfrak{D}_1$ does not descend from $\ddot{\mathfrak{Y}}$ to $\mathfrak{Y}$. It is also the $\frac{1}{2}$ in the formula for the theta function...

The nonarchimedean theta function (after Mumford)

Let $\mathfrak{U}$ be the open formal subscheme of $\mathfrak{Y}$ missing the nodes on component 0 of the special fiber. Tate uniformization (and the choice of orientation) defines a multiplicative coordinate $U \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}}^\times)$ admitting a square root $\ddot{U}$ on $\ddot{\mathfrak{U}} = \mathfrak{U} \times_{\mathfrak{Y}} \ddot{\mathfrak{Y}}$.

On $\ddot{\mathfrak{Y}}$, we have a meromorphic function given on $\ddot{\mathfrak{U}}$ by the formula

$$\ddot{\Theta}(\ddot{U}) = q^{-\frac{1}{8}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} \ddot{U}^{2n+1}.$$

Its zero divisor is $1\times$(cusps); its pole divisor in $\mathfrak{D}_1$. We have

$$\ddot{\Theta}(\ddot{U}) = -\ddot{\Theta}(\ddot{U}^{-1}), \qquad \ddot{\Theta}(-\ddot{U}) = -\ddot{\Theta}(\ddot{U}),$$
$$\ddot{\Theta}(q^{a/2}\ddot{U}) = (-1)^a q^{-a^2/2} \ddot{U}^{-2a} \ddot{\Theta}(\ddot{U}).$$

Étale theta classes and the theta function

Proposition (EtTh, Proposition 1.4)

The étale theta classes $\mathfrak{o}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta} \in H^1(\pi_1^{\mathrm{tm}}(\ddot{Y}), \Delta_{\Theta})$ *agree with the Kummer classes associated to* $\mathfrak{o}^{\times}_{\ddot{K}}$*-multiples of* $\ddot{\Theta}$ *as a regular function on* $\ddot{Y}$ *(as in Stix's lecture).*

In particular, for $L/\ddot{K}$ *finite and* $y \in \ddot{Y}(L)$ *not cuspidal, the restricted classes*

$$\mathfrak{o}^{\times}_{\ddot{K}} \cdot \ddot{\eta}^{\Theta}|_y \in H^1(G_L, \Delta_{\Theta} \cong \widehat{\mathbb{Z}}(1)) \cong (L^{\times})^{\wedge}$$

lie in $L^{\times}$ *and equal the* $\mathfrak{o}^{\times}_{\ddot{K}}$*-multiples of the value* $\ddot{\Theta}(y)$*. (There is a similar statement also for cusps.)*

For this reason, the étale theta classes are also referred to as the *étale theta function*.

Rigidity and indeterminacies: what?

Say we have two collections of data as above, distinguished by the subscripts α, β.

For IUT, one needs various statements saying that certain constructions are *rigid*. That is, given an isomorphism of certain topological groups (or analogous data, such as Frobenioids), we reconstruct an isomorphism of certain underlying arithmetic-geometric structures...

... but only up to some specified *indeterminacy*. That is, we generally only recover a collection of closely related isomorphisms.

Rigidity and indeterminacies: why?

In Bogomolov-Zhang, we made a certain calculation in the ambient group $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$. In that calculation, each γ_s in isolation is only defined up to conjugation, but the global geometry allows us to view the collection of elements $\alpha_i, \beta_i, \gamma_s$ as itself being *synchronized*, i.e., well-defined up to a single *overall* conjugation.

In IUT, one does not start with an ambient group. Instead, one feeds in the *tempered Frobenioids*, whose structure reflects the original geometry only via rigidity. Indeterminacies correspond to the effect of outer conjugations from the (nonexistent) ambient group.

Eventually, one must *realize*[5] the interactions among various data. Indeterminacies are to be reflected in *volumes*; containments then give meaningful Diophantine inequalities.

This is (?) like computing with real numbers using *interval arithmetic*.

[5]= convert into subsets of some $\mathbb{R}^n$

Example: tempered anabelian rigidity

Theorem (EtTh, Theorem 1.6)

Let $\gamma : \pi_1^{\mathrm{tm}}(X_\alpha) \cong \pi_1^{\mathrm{tm}}(X_\beta)$ *be an isomorphism of topological groups.*

(i) *We have* $\gamma(\pi_1^{\mathrm{tm}}(\ddot{Y}_\alpha)) = \pi_1^{\mathrm{tm}}(\ddot{Y}_\beta)$.

(ii) *The map* γ *induces an isomorphism* $\Delta_{\Theta,\alpha} \cong \Delta_{\Theta,\beta}$ *compatible with the Kummer+valuation maps*

$$H^1(G_{\ddot{K}_*}, \Delta_{\Theta,*} \cong \widehat{\mathbb{Z}}(1)) \cong (\ddot{K}_*^\times)^\wedge \to \widehat{\mathbb{Z}} \qquad (* = \alpha, \beta).$$

(iii) *The map* γ *induces an isomorphism of cohomology carrying*

$$\mathfrak{o}_{\ddot{K}_\alpha}^\times \cdot \ddot{\eta}_\alpha^\Theta \in H^1(\pi_1^{\mathrm{tm}}(\ddot{Y}_\alpha), \Delta_{\Theta,\alpha})$$

to a $\mathrm{Gal}(Y_\beta/X_\beta) \cong \mathbb{Z}$*-conjugate of the corresponding classes for* β.

Example: tempered anabelian rigidity (discussion)

Much of the content of this statement is from "Semi-graphs of Anabelioids" (see the lectures of Szamuely, Lepage).

In part (iii) of the previous theorem, it is only immediately obvious that the classes agree after extending scalars from $\mathfrak{o}_{\ddot{K}}^{\times}$ to $\ddot{K}^{\times}$. The reduction of indeterminacy uses the compatibility with the theta function at the cusps, where one can use the canonical integral structure.

Are the evaluations at nonzero torsion points relevant for the global theory?[6] Better yet, can they be used to "concretize" some of IUT?

[6] Response from Mochizuki: the global nature of torsion points is incorporated into the construction of Hodge theaters, in a manner analogous to the role played in Bogomolov-Zhang by the cusps $\mathbb{P}^1(\mathbb{Q})$ for the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane.

Even more Galois coverings

Hereafter, fix an odd prime ℓ, and assume[7] $p > 2$, $K = \ddot{K}$, and $\ell \nmid v_K(q)$.

Proposition (EtTh, Proposition 2.2)

(i) *The group $E[\ell](K)$ is of order ℓ. Let $\underline{X} \to X$ be the corresponding $\mathbb{Z}/\ell\mathbb{Z}$-cover (with K-rational cusps).*

(ii) *There is a unique $\mathbb{Z}/\ell\mathbb{Z}$-cover $\underline{\underline{X}}$ of $\underline{X}$ whose Galois group covers the -1 eigenspace of multiplication by -1 on $\widehat{\pi}_1(\underline{X})^{\mathrm{ell}}$.*

Put $C = X/\pm 1$, $\underline{C} = \underline{X}/\pm 1$; then $\underline{\underline{X}} \to \underline{X}$ is the pullback of a double cover $\underline{\underline{C}} \to \underline{C}$.

For any cover W of X, let $\underline{\underline{W}}$ be the composite of W with $\underline{\underline{C}}$ over C.

[7] It is probably safer to also assume $p \neq \ell$.

Cyclotomic envelopes

For $\Pi \twoheadrightarrow G_K$ a surjection of topological groups, define

$$\Pi[\mu_N] := \Pi \times_{G_K} (\mathbb{Z}/N\mathbb{Z}(1) \rtimes G_K).$$

There is a tautological surjection $s^{\text{taut}} : \Pi \to \Pi[\mu_N]$ (also called the *algebraic section*).

For $\Delta = \ker(\Pi \to G_K)$, also write

$$\Delta[\mu_N] := \ker(\Pi[\mu_N] \twoheadrightarrow G_K).$$

Theta environments

The étale theta classes $\mathcal{O}_K^\times \cdot \ddot{\eta}^\Theta$ give rise (with some effort) to a $(\mathcal{O}_K^\times)^\ell$-multiple of classes in $H^1(\pi_1^{\mathrm{tm}}(\ddot{Y}), \Delta_\Theta/\ell)$, and then to classes

$$\underline{\underline{\ddot{\eta}}}^\Theta \in H^1(\pi_1^{\mathrm{tm}}(\underline{\underline{\ddot{Y}}}), \ell\Delta_\Theta).$$

The *(mod N) model mono-theta environment* associated to X consists of these data:

(a) the topological group $\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N]$;

(b) the subgroup of $\mathrm{Out}(\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N])$ generated by $\mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ and

$$K^\times \to (K^\times)/(K^\times)^N \cong H^1(G_K, \mu_N)$$
$$\hookrightarrow H^1(\pi_1^{\mathrm{tm}}(\underline{\underline{Y}}), \mu_N) \to \mathrm{Out}(\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N]);$$

(c) the μ_N-conjugacy class of subgroups of $\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N]$ containing the image of the *theta section* $s^\Theta_{\underline{\underline{\ddot{Y}}}} = s^{\mathrm{taut}}_{\underline{\underline{\ddot{Y}}}}$ - (any class in the $(\ell\,\mathrm{Gal}(Y/X) \times \mu_2)$-orbit of $\underline{\underline{\ddot{\eta}}}^\Theta$).

Theta environments (continued)

The *(mod N) model bi-theta environment* X is the same plus:

(d) the μ_N-conjugacy class of subgroups of $\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N]$ containing the image of the tautological section.

A *mono/bi-theta environment* is a set of data abstractly isomorphic to a model mono/bi-theta environment.

Rigidity properties

Theorem (EtTh, Corollary 2.19)

Take a model mono-theta environment associated to X.

(a) **Cyclotomic rigidity:** *One reconstructs*[a] *the subquotients* $(\ell\Delta_\Theta)[\mu_N] \subseteq \pi_1^{\mathrm{tm}}(\underline{\underline{Y}}_{\overline{K}})^\Theta[\mu_N] \subseteq \pi_1^{\mathrm{tm}}(\underline{\underline{Y}})^\Theta[\mu_N]$ *of* $\pi_1^{\mathrm{tm}}(\underline{\underline{Y}})[\mu_N]$*, and the two splittings of* $(\ell\Delta_\Theta)[\mu_N] \twoheadrightarrow \ell\Delta_\Theta$ *determined by the tautological and theta sections.*

(b) **Discrete rigidity**: *Any "abstract" projective system formed by the mod N mono-theta environments is isomorphic to the "standard" one.*

(c) **Constant multiple rigidity:** *Assume $\sqrt{-1} \in K$ and (...)*[b]*. From $\pi_1^{\mathrm{tm}}(\underline{\underline{X}})$, one reconstructs the $(\ell\,\mathrm{Gal}(Y/X) \times \mu_2)$-orbit of $\mu_\ell \cdot \underline{\underline{\ddot{\eta}}}^\Theta$. In particular, using (a), any projective system of mono-theta environments promotes to a system of bi-theta environments.*

---

[a] via a "functorial group-theoretic algorithm". Definable sets, anyone?

[b] A normalization condition on η^Θ on X_N (but not ℓ) that I couldn't parse.

Frobenioids: the standard geometric example

Recall the geometric example from Ben-Bassat's, Czerniawska's lectures.

Let V be a proper normal geometrically integral variety over a field k, with function field K. Fix a collection $\mathbb{D}_K$ of $\mathbb{Q}$-Cartier prime divisors on L.

Let $\mathcal{B}(G_K)$ be the (connected) Galois category. For $L \in \mathcal{B}(G_K)$, let $V[L]$ be the normalization of V in L. Let $\mathbb{D}_L$ be the set of prime divisors of $V[L]$ which map into $\mathbb{D}_K$; assume these are all $\mathbb{Q}$-Cartier.

Consider the category of pairs $(L, \mathcal{L})$ with $L \in \mathcal{B}(G_K)$ and $\mathcal{L}$ a line bundle on $V[L]$ represented by a Cartier divisor supported in $\mathbb{D}_L$. A morphism $(L, \mathcal{L}) \to (M, \mathcal{M})$ consists of:

- a morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(M)$;
- an integer $d \geq 1$;
- a morphism $\mathcal{L}^{\otimes d} \to \mathcal{M}|_{V[L]}$ whose zero locus is a Cartier divisor supported in $\mathbb{D}_L$.

The tempered Frobenioid [EtTh, Example 3.9]

For W a covering of X (or C), let $\mathcal{D}_W$ be the Galois category (temperoid) of connected tempered covers of W. Let $\mathcal{D}_W^{\mathrm{ell}}$ be the subcategory of covers unramified over cusps; then $\mathcal{D}_W^{\mathrm{ell}} \hookrightarrow \mathcal{D}_W$ admits a left adjoint.

Define a functor $\Phi_{W,0}$ from $\mathcal{D}_W$ to monoids:

$$\Phi_{W,0}(W') := \varprojlim_{W''/W' \text{ Galois}} \mathrm{Div}_+(W'')^{\mathrm{Gal}(W''/W')}.$$

Let Φ_W be the perfection (divisible closure) of Φ_0.

For $W' \in \mathcal{D}_W^{\mathrm{ell}}$, let $\Phi_W^{\mathrm{ell}} \subseteq \Phi_W$ be the perf-saturation of the submonoid of $\Phi_{W,0}$ where W'' only runs over covers for which $W'' \to W' \to W$ is the universal topological cover of some finite étale cover of W (e.g., $Y \to X$).

For $\alpha : W'' \to W'$ a morphism of coverings, view α as an object of $(\mathcal{D}_W)_{W'}$ (objects over W') and put $\mathcal{D}_\alpha := (\mathcal{D}_W)_{W'}[\alpha]$ (objects over α).

Rigidity [EtTh, §5; IUT, Example 3.2]

The following statements (in precise forms) play key roles in IUT (see Mok's lecture and subsequent).

Proposition (EtTh, Corollary 3.8)

The p-adic Frobenioid is reconstructed from the tempered Frobenioid.

Theorem (EtTh, Theorem 5.7)

The ℓ-th root étale theta classes are reconstructed (up to $\ell\mathbb{Z} \times \mu_{2\ell}$-indeterminacy) from the tempered Frobenioid.

Theorem (EtTh, Theorem 5.10)

The rigidities of mono-theta environments (cyclotomic, discrete, constant multiple) can be asserted in the language of Frobenioids.