Étale and crystalline companions

Kiran S. Kedlaya

Department of Mathematics, University of California, San Diego*
kedlaya@ucsd.edu
http://kskedlaya.org/slides/

Algebraic and arithmetic geometry seminar
Università di Pisa (virtual)
September 30, 2020

Kedlaya was supported by NSF (grant DMS-1802161 and prior grants), UC San Diego (Warschawski Professorship), and IAS (Visiting Professorship 2018–2019).

*The UCSD campus sits on the ancestral homelands of the Kumeya'ay Nation; the Kumeya'ay people continue to have an important and thriving presence in the region.

Zeta functions of algebraic varieties

Let $\mathbb{F}_q$ denote a finite field of order $q = p^a$. For X an algebraic variety over $\mathbb{F}_q$, the <u>zeta function</u> of X is

$$Z(X,T) = \prod_{x \in X^\circ} (1 - T^{\deg(x/\mathbb{F}_q)})^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right),$$

where X° denotes the closed points of X (i.e., Galois orbits of $\overline{\mathbb{F}}_q$-points).

- For $X = \mathbb{A}^n_{\mathbb{F}_q}$ (affine space), $Z(X,T) = \frac{1}{1-q^nT}$.
- For $X = V \sqcup W$ (disjoint union), $Z(X,T) = Z(V,T)Z(W,T)$.
- For $X = \mathbb{P}^n_{\mathbb{F}_q}$ (projective space), $Z(X,T) = \frac{1}{(1-T)(1-qT)\cdots(1-q^nT)}$.
- For X an elliptic curve, $Z(X,T) = \frac{1-aT+qT^2}{(1-T)(1-qT)}$ with $a \in \mathbb{Z} \cap [-2\sqrt{q}, 2\sqrt{q}]$ (Hasse).

The Weil conjectures

Theorem

Let X be a scheme of finite type over $\mathbb{F}_q$.

- *$Z(X, T)$ is a rational function of T.*
- *Suppose in addition that $X/\mathbb{F}_q$ is smooth proper of dimension d. Then*
$$Z(X, T) = \frac{P_1(T)\dots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)}$$
where $P_i(T) \in 1 + T\mathbb{Z}[T]$ has roots in $\mathbb{C}$ on the circle $|T| = q^{-i/2}$. Moreover, $P_{d+i}(T) = P_{d-i}(q^i T)$.
- *Suppose in addition that X lifts to a smooth proper scheme $\tilde{X}$ over some valuation ring R of characteristic 0. Then for any embedding $R \hookrightarrow \mathbb{C}$, $\deg(P_i)$ equals the i-th Betti number of $(\tilde{X}_{\mathbb{C}})^{\mathrm{an}}$.*

Weil cohomology

Weil proposed to interpret these statements in terms of a hypothetical **Weil cohomology**, associating to X a collection of finite-dimensional K-vector spaces. Rationality of $Z(X, T)$ would follow from the **Lefschetz trace formula** for Frobenius:

$$\#X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2d} (-1)^i \operatorname{Trace}(\mathrm{Frob}_X^n, H^i(X)).$$

The functional equation would follow from Poincaré duality. The comparison with Betti numbers would follow from a comparison isomorphism with some cohomology theory for complex manifolds. The Riemann hypothesis "should" follow from an (unknown) form of the Hodge index theorem on $X \times_{\mathbb{F}_q} X$.

Two candidates

In this talk, we contrast two constructions of Weil cohomology. Note that the first is actually infinitely many constructions one for each prime $\ell \neq p$,

- **Étale cohomology**: inspired by fundamental groups and Betti (co)homology. Here $K = \overline{\mathbb{Q}}_\ell$ for some prime $\ell \neq p$.
- **Crystalline/rigid cohomology**: inspired by differential forms and de Rham cohomology. Here $K = \overline{\mathbb{Q}}_p$.

Our main result asserts a form of "interchangeability" between these approaches. This is important because these two constructions have different strengths and weaknesses; étale cohomology can be defined in greater generality (e.g., on arithmetic schemes) while crystalline/rigid cohomology is strongly linked to other geometric constructions (and is often better for explicit computations).

Coefficient objects

In both étale and crystalline/rigid cohomology, one has a theory of cohomology valued in a "local system" satisfying certain formal properties (e.g., Leray spectral sequence). More on what these are later.

In particular, if $f : Y \to X$ is smooth proper and $\mathcal{O}_Y$ is the trivial local system on Y, it admits higher direct images $R^i f_* \mathcal{O}_Y$ on X. For each $x \in X^\circ$, we can restrict $R^i f_* \mathcal{O}_Y$ to x to get a finite-dimensional K-vector space with a K-linear action of Frob_x; the reverse charpoly is the factor P_i of $Z(f^{-1}(x), T)$.

A question of Deligne

Deligne asked whether **all** local systems have "geometric origin", in the sense that they can be generated (by taking subquotients, extensions, and twists) from ones arising from smooth proper morphisms as above.

Deligne observed that for $\dim(X) = 1$, this would follow from the Langlands correspondence for $\mathrm{GL}(n)$ (which is now known). However, in general we have no idea how to prove this!

We prove something less: given two categories of local systems and an identification of $\overline{\mathbb{Q}}$ within each field, for any local system in one category whose charpolys are all defined over $\overline{\mathbb{Q}}$, there is a local system in the other category with the same charpolys at all closed points. (That is, these two form part of a **compatible system** of coefficient objects.)

Fundamental groups in algebraic geometry

A key construction in étale cohomology is the **fundamental group** associated to (connected) X and a geometric base point $\overline{x}$. This exploits the analogy between deck transformations of covering spaces and automorphisms of field extensions; however, in algebraic geometry we can only handle **finite** covering space maps (a/k/a finite étale surjections), so we only get a **profinite** étale fundamental group $\pi_1(X, \overline{x})$.

When $X = \{x\}$ is a single closed point, $\pi_1(X, \overline{x})$ is the absolute Galois group of $\kappa(x)$. This group is topologically generated by Frob_x.

In general, an open subgroup G of $\pi_1(X, \overline{x})$ can be interpreted as $\pi_1(Y, \overline{x})$ for some finite étale covering $Y \to X$ through which $\overline{x}$ factors.

Local systems in algebraic geometry

For $\ell \neq p$ prime, by an **ℓ-adic local system** on X, I will mean a continuous representation $\mathcal{E}$ of $\pi_1(X)$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.

For any morphism $Y \to X$ of varieties, I get a morphism $\pi_1(Y) \to \pi_1(X)$, so I can pull back an ℓ-adic local system on X to an ℓ-adic local system on Y. In particular, if $Y = \{x\}$ is a closed point, then I get a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with $G_{\kappa(x)}$-action; let $P(\mathcal{E}, x)$ be the characteristic polynomial of the action of (geometric) Frobenius.

One can define **étale cohomology** with coefficients in a local system by considering the **étale topology**, a Grothendieck topology combining the Zariski topology with unramified (étale) maps.

What goes wrong for $\ell = p$?

One can define étale cohomology with ℓ-adic coefficients even for $\ell = p$, but this cannot be used in the same way to understand zeta functions.

A basic example of the key difficulty: if X is an elliptic curve over $\mathbb{F}_q$, then

$$X[\ell](\overline{\mathbb{F}}_q) \cong \begin{cases} (\mathbb{Z}/\ell\mathbb{Z})^2 & (\ell \neq p) \\ 0 \text{ or } \mathbb{Z}/\ell\mathbb{Z} & (\ell = p). \end{cases}$$

Consequently, if one computes the first étale cohomology with p-adic coefficients, it has dimension 0 or 1 instead of 2, and thus fails to match with the corresponding Betti number.

Crystalline vs. rigid cohomology

Dwork originally proved the rationality of $Z(X,T)$ using p-adic analysis. This did not explicitly use cohomology.

Based on Grothendieck's infinitesimal site, Berthelot introduced **crystalline cohomology**: this is an integral p-adic cohomology which behaves well for smooth proper schemes over $\mathbb{F}_q$.

Meanwhile, Monsky and Washnitzer developed a rational p-adic cohomology for smooth affine schemes over $\mathbb{F}_q$; Berthelot globalized this to obtain **rigid cohomology**.

The p-adic analogue of a local system

For X (smooth) affine, one can define† **overconvergent F-isocrystals** and their cohomology in terms of a smooth lift $\mathfrak{X}$ of X over some finite extension of $\mathbb{Z}_p$.

For concreteness, take the example $X = \mathbb{A}^n_{\mathbb{F}_q}$. Let K be a finite extension of $\mathbb{Q}_p$ with residue field containing $\mathbb{F}_q$ and take $\mathfrak{X} = \mathbb{A}^n_{\mathfrak{o}_K}$. The Raynaud generic fiber of $\mathfrak{X}$ is the closed unit n-disc over K; take a vector bundle $\mathcal{E}$ on it equipped with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}$. This approximately‡ defines a **convergent**§ **isocrystal** on X.

To add F, pick a morphism $\sigma : \mathfrak{X} \to \mathfrak{X}$ lifting the absolute Frobenius map $x \mapsto x^p$ on X (e.g., in this example, take each coordinate to its p-th power). Then a **convergent F-isocrystal** is a convergent isocrystal $\mathcal{E}$ together with an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ that respects connections.

†One has to work to show that the definition doesn't depend on anything but X.

‡Modulo a convergence condition on Taylor series which is forced by Frobenius.

§To get an **overconvergent** isocrystal, one must extend to a slightly larger disc.

A useful analogy

There is an exact sequence

$$1 \to \pi_1(X_{\overline{\mathbb{F}}_q}) \to \pi_1(X) \to G_{\mathbb{F}_q} \to 1.$$

The group $\pi_1(X_{\overline{\mathbb{F}}_q})$ is often easier to describe (e.g., for a projective curve of genus g it is profinite on $2g$ generators with one relation) and has many representations which cannot be extended to $\pi_1(X)$; the interaction with $G_{\mathbb{F}_q}$ imposes severe constraints.

Analogously, there exist lots of (over)convergent isocrystals, because they can be described using only one structure (an integrable connection); however, (over)convergent F-isocrystals are specified using two separate structures which must be compatible in a nontrivial way.

Cohomology of a p-adic local system

Given a convergent F-isocrystal $\mathcal{E}$, the integrability condition means that there is a **de Rham complex**

$$0 \to \mathcal{E} \stackrel{\nabla}{\to} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K} \stackrel{\nabla^{(2)}}{\to} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/K} \to \cdots$$

where $\nabla^{(2)}(f \otimes dg) = \nabla(f) \wedge dg$. The cohomology[¶] of this complex is the **rigid cohomology of X with coefficients in $\mathcal{E}$** (or for short, the cohomology of $\mathcal{E}$). These are finite-dimensional K-vector spaces (K, 2006).

To compare with ℓ-adic cohomology, we want a K-linear action of F, but right now this is off by an automorphism of K. We fix this by tensoring over $\mathbb{Q}_p$ with $\overline{\mathbb{Q}}_p$ and taking fixed subspaces for the automorphism of K (acting trivially on $\overline{\mathbb{Q}}_p$); this gives an $\overline{\mathbb{Q}}_p$-vector space with linear F-action.

The same applies to the restriction to a closed point x, giving us the Frobenius characteristic polynomial $P(\mathcal{E}, x)$.

[¶]If X were not affine, we would have to take hypercohomology instead.

Motivation

An ℓ-adic or p-adic local system $\mathcal{E}$ on X **comes from geometry** if it appears in the higher direct images of some smooth proper morphism $f : Y \to X$. Concretely, this means that for each $x \in X^\circ$, $P(\mathcal{E}, x)$ shows up as a factor in the zeta function of the fiber of f over x.

As a consequence of the Weil conjectures, ℓ-adic and p-adic local systems that come from geometry have very strong arithmeticity properties. Deligne conjectured[‖] that every ℓ-adic local system on X looks like it "comes from geometry", up to separating into irreducibles and taking twists.

To rigidify the situation enough to formulate the conjecture, we assume that $\mathcal{E}$ is irreducible and that its determinant is a character of finite order; this second restriction eliminates most twists.

[‖] See "La conjecture de Weil, II" (1979), Conjecture 1.2.10.

Historical note

Crew was the first to suggest that overconvergent F-isocrystals should be the appropriate p-adic analogue of ℓ-adic local systems.

Since the mid-1980s, it has been expected that rigid cohomology obeys formal properties analogous to those of étale cohomology, to the extent that (for example) it could be used to rederive the Weil conjectures. These formal properties are now known, thanks to difficult work by many authors (Abe, Berthelot, Chiarellotto, Caro, Crew, de Jong, K, Shiho, Tsuzuki).

The theorem we state on the next slide confirms the modified form of Deligne's conjecture which treats ℓ-adic and p-adic local systems on an equal footing.

Statement of the main theorem

Theorem (L. Lafforgue, Abe, Deligne, Drinfeld, Abe–Esnault, K)

Let $\mathcal{E}$ be an ℓ-adic local system on X (allowing $\ell = p$). Assume that $\mathcal{E}$ is irreducible and its determinant is of finite order.

(i) *The roots of $P(\mathcal{E}, x)$ in $\overline{\mathbb{Q}}_\ell$ are all algebraic over $\mathbb{Q}$, and their conjugates in $\mathbb{C}$ all lie on the unit circle.*

(ii) *The coefficients of $P(\mathcal{E}, x)$ lie in a number field depending only on $\mathcal{E}$.*

(iii) *The roots of $P(\mathcal{E}, x)$ are p-units (i.e., integral over $\mathbb{Z}[p^{-1}]$).*

(iv) *For any p-adic valuation v on $\overline{\mathbb{Q}}$, the roots of $P(\mathcal{E}, x)$ all have valuation in the range $[-\frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x)), \frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x))]$.*

(v) *Fix a prime $\ell' \neq p$ and an isomorphism of $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$ with $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell'}$. Then there exists an ℓ'-adic local system $\mathcal{E}'$ on X such that $P(\mathcal{E}, x) = P(\mathcal{E}', x)$ for all $x \in X^\circ$. (We say $\mathcal{E}'$ is a* companion *of $\mathcal{E}$.)*

(vi) *The analogue of (v) holds for $\ell' = p$.*

The Langlands correspondence for function fields

Let X be a curve over $\mathbb{F}_q$, with smooth compactification $\overline{X}$. Then $\mathbb{F}_q(X)$ is a function field of transcendence degree 1, and as such is strongly analogous to a number field. In particular, we may define its adèle ring $\mathbb{A}_{\mathbb{F}_q(X)}$ by taking a restricted product of its completions at all places.

For any $\ell \neq p$, Langlands conjectured** a correspondence between rank-n irreducible ℓ-adic local systems with determinant of finite order and cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{F}_q(X)})$ unramified on X, in which $P(\mathcal{E}, x)$ matches the characteristic polynomial of a Hecke operator attached to x.

For $n = 1$, this reproduces class field theory for $\mathbb{F}_q(X)$. For $n = 2$, this was shown by Drinfeld; for $n > 2$, this was done by L. Lafforgue.

One also has a similar correspondence involving overconvergent F-isocrystals. This was shown by T. Abe, emulating Lafforgue.

**This is only the GL_n case of the conjecture. For more general reductive groups, the automorphic-to-geometric direction has recently been constructed by V. Lafforgue.

The Langlands correspondence and geometric origins

As stated, the Langlands correspondence is not enough information for our purposes. However, the proof gives much more: it shows that ℓ-adic local systems and overconvergent F-isocrystals come from geometry, specifically from moduli spaces of **shtukas** (certain vector bundles with extra structure); these act like Shimura varieties.

Combining this with the Langlands bijection, the matching of characteristic polynomials (plus a similar statement about points of $\overline{X} \setminus X$), and properties of étale and rigid cohomology, one gets a complete resolution of Deligne's (updated) conjecture when X is a curve.

Goodbye Langlands correspondence... and hello again

For X of dimension > 1, there is not even a conjectural analogue of the Langlands correspondence that would provide geometric origins for local systems. Even the analogue of class field theory is quite subtle.

Instead, we use the case of curves as a black box. For every curve in X, we can use the correspondence to gather information; in addition, when two curves cross at x, the values of $P(\mathcal{E}, x)$ on the two curves coincide.

Parts (i), (iii), (iv) of the theorem involve valuations of roots of $P(\mathcal{E}, x)$ at individual points, so this is essentially enough; one onlyneeds to check that $\mathcal{E}$ "usually" remains irreducible after restricting to a curve.

For the remaining components, more work is required; the general theme is to exploit **uniformity** statements quantified over curves in X.

Deligne's finiteness argument

Part (ii) states that the $P(\mathcal{E}, x)$ all have coefficients in a single number field. For $\ell \neq p$, Deligne (2012) shows this by showing that $\mathcal{E}$ is uniquely determined (up to semisimplification) by its $P(\mathcal{E}, x)$ for x of degree up to some explicit bound. (By Galois theory, the field generated by the coefficients of those $P(\mathcal{E}, x)$ contains all the rest.)

This can be done by restricting to curves, making sure that the cutoff value can be chosen uniformly. This case is treated by a direct calculation involving L-functions. A key point is that by replacing X with some finite étale cover, one can kill all wild ramification of $\mathcal{E}$, and thus control the Euler characteristic by the Grothendieck–Ogg–Shafarevich formula.

This argument adapts easily to the case $\ell = p$, modulo the "key point": for $\ell \neq p$, this is handled by choosing a lattice in the associated representation and trivializing it mod ℓ; any remaining ramification is of ℓ-power order and hence tame. For $\ell = p$, something similar is true but deep (K, 2011).

Drinfeld's patching construction

Part (v) involves constructing an ℓ'-adic companion of $\mathcal{E}$. Since such an object is a representation of $\pi_1(X)$, we can try to do it by building a compatible family of mod-$(\ell')^n$ representations.

For $\ell \neq p$, Drinfeld (2012) proves this by interpolating this family from the restrictions to curves, where we have such families using the Langlands correspondence. The key point is that modulo any fixed power of ℓ', one can build a **finite** family of representations such that for any curve, one of these has the right restriction. Then an easy compactness argument lets you put together the compatible family.

It is relatively easy to adapt this to the case where $\mathcal{E}$ is an overconvergent F-isocrystal, since we are still assuming $\ell' \neq p$. This was carried out (independently) by Abe–Esnault and K (2017).

A p-adic analogue of Drinfeld's construction

Part (vi) of Deligne's conjecture involves constructing a p-adic companion of $\mathcal{E}$. This is not a straightforward adaptation of the ℓ-adic case, because overconvergent F-isocrystals cannot be described in terms of representations of $\pi_1(X)$ (except in some trivial cases). However, one can still execute a form of Drinfeld's strategy (K, 2020).

It suffices[††] to treat the case where $X \to S$ is a stable curve fibration. We define moduli stacks of "mod-p^n truncated crystals on X"; these are coherent sheaves on certain schemes built from X using Witt vecors. Crucially, these are of **finite type** over $\mathbb{F}_q$; this follows from a uniformity property for Chern classes of isocrystals on curves.

[††] This reduction leans heavily on being able to go from $\ell = p$ back to $\ell \neq p$, which we already handled.

A compactness argument in moduli stacks

We have a tower of qcqs Artin stacks over X:

$$\cdots \to M_{n+1} \to M_n \to \cdots \to M_1$$

in which the transition maps are ~~proper~~ universally closed.

On each smooth fiber of $X \to S$, we have **companion points** arising from a crystalline companion on the fiber. These must accumulate on some irreducible component.

By Chebotarev density[‡‡], **every** point in this component is a companion point. Moreover, every fiber of $M_n \to X$ consists of isolated points with the same residue field (and no stack structure). We deduce that $M_n \to X$ is a disjoint union of isomorphs of X.

[‡‡]This only works on "typical" fibers, and a bootstrapping argument is needed to finish. Due to the differences between convergent and overconvergent isocrystals, we must finesse some issues using Tsuzuki's **minimal slope theorem**.

The main theorem again

Theorem (L. Lafforgue, Abe, Deligne, Drinfeld, Abe–Esnault, K)

Let $\mathcal{E}$ be an ℓ-adic local system on X (allowing $\ell = p$). Assume that $\mathcal{E}$ is irreducible and its determinant is of finite order.

(i) *The roots of $P(\mathcal{E}, x)$ in $\overline{\mathbb{Q}}_\ell$ are all algebraic over $\mathbb{Q}$, and their conjugates in $\mathbb{C}$ all lie on the unit circle.*

(ii) *The coefficients of $P(\mathcal{E}, x)$ lie in a number field depending only on $\mathcal{E}$.*

(iii) *The roots of $P(\mathcal{E}, x)$ are p-units (i.e., integral over $\mathbb{Z}[p^{-1}]$).*

(iv) *For any p-adic valuation v on $\overline{\mathbb{Q}}$, the roots of $P(\mathcal{E}, x)$ all have valuation in the range $[-\frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x)), \frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x))]$.*

(v) *Fix a prime $\ell' \neq p$ and an isomorphism of $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$ with $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell'}$. Then there exists an ℓ'-adic local system $\mathcal{E}'$ on X such that $P(\mathcal{E}, x) = P(\mathcal{E}', x)$ for all $x \in X^\circ$. (We say $\mathcal{E}'$ is a* companion *of $\mathcal{E}$.)*

(vi) *The analogue of (v) holds for $\ell' = p$.*

Newton polygons

Let v be a p-adic valuation on $\overline{\mathbb{Q}}_p$. For each $x \in X^\circ$, let $N(\mathcal{E}, x, v)$ denote the **(normalized) Newton polygon** of $P(\mathcal{E}, x, v)$; that is, it is the graph of a convex, piecewise linear function from $[0, \operatorname{rank}(\mathcal{E})]$ to $\mathbb{R}$ which has one length-1 interval of slope $v(\alpha)/v(\#\kappa(x))$ for each root α of $P(\mathcal{E}, x)$. (Part (iv) asserts that these slopes are at most $\frac{1}{2}\operatorname{rank}(\mathcal{E})$.)

For $\mathcal{E}$ an overconvergent F-isocrystal, the function $x \mapsto N(\mathcal{E}, x, v)$ is upper semicontinuous (Grothendieck–Katz); in particular, it takes only finitely many values, and each value occurs on a locally closed stratum. Moreover, jumps only occur in codimension 1 (de Jong–Oort).

For $\mathcal{E}$ an ℓ-adic local system, the only way we know how to prove an analogous statement is via (vi).

Concrete example: if E is an elliptic curve over $\mathbb{F}_q$ with $q = p^n$ and $a_q = q + 1 - \#E(\mathbb{F}_q)$, then a_q is divisible by p iff it is divisible by $p^{\lfloor n/2 \rfloor}$.

p-adic companions and L-functions

To any ℓ-adic local system $\mathcal{E}$, we may define its associated L-function

$$L(\mathcal{E}, T) = \prod_{x \in X^\circ} \det(1 - FT, \mathcal{E}_x)^{-1};$$

it is a rational function of T. Note that $\det(1 - FT, \mathcal{E}_x)$ is the reverse of the polynomial $P(\mathcal{E}, x)$; consequently, Deligne's conjecture implies statements about the zeroes/poles of $L(\mathcal{E}, T)$ analogous to the Weil conjectures.

If $\mathcal{E}$ admits a p-adic companion, we can also analyze the **unit-root** L**-function**, in which (for some p-adic valuation) we retain only the factor of $\det(1 - FT)$ consisting of roots of valuation 0; this is not typically a rational function, but is a ratio of two p-adically entire power series (Dwork's conjecture, proved by Wan).

Geometric origins

In some cases, the existence of a p-adic companion can be used to establish that an ℓ-adic local system comes from geometry. Specifically, one can show that certain rank-2 local systems arise from families of abelian varieties (Krishnamoorthy-Pál). This uses the p-adic companion to construct Dieudonné crystals; this is a p-adic analogue of a Hodge-theoretic argument of Corlette–Simpson.