Beyond Pick's theorem: Ehrhart polynomials and mixed volumes

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Contents

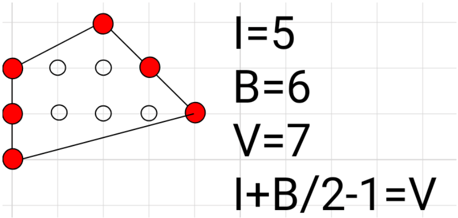
Pick's* theorem

Theorem (Pick, 1899)

Let P be a polygon in $\mathbb{R}^2$ with vertices at lattice points (elements of $\mathbb{Z}^2$).

- *Let V be the area of (the interior of) P.*
- *Let I be the number of lattice points in the interior of P.*
- *Let B be the number of lattice points on the boundary of P.*

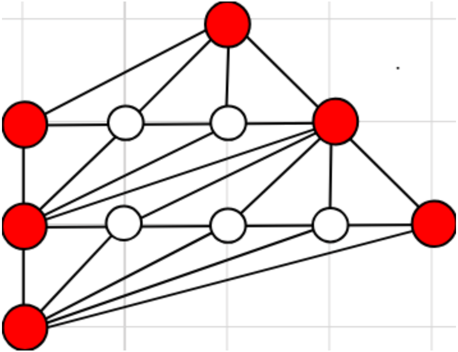
Then $V = I + \frac{1}{2}B - 1$.



*Pick died in a Nazi murder camp in 1942 without having received much recognition for this theorem; it was popularized by Steinhaus in the 1960s.

A proof of Pick's theorem (part 1)

One can reduce Pick's theorem to the case of a triangle with no interior lattice points: one can always dissect P into some such triangles, and both sides of the formula are additive.

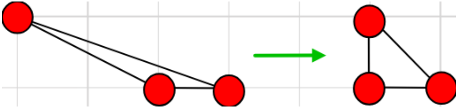


A proof of Pick's theorem (part 2)

Let T be a lattice triangle with no interior lattice points. Using continued fractions†, one can find a matrix in $\mathrm{SL}_2(\mathbb{Z})$ which transforms T into the standard lattice triangle with vertices

$$(0,0),(0,1),(1,0).$$

Both sides of Pick's formula are invariant under this transformation, and the equality for the standard triangle is easy to check.



†Ultimately this means Euclid's algorithm (300 BCE), but continued fractions don't appear in a "modern" form until the *Āryabhaṭīyaṃ* (500 CE).

Contents

Lattice polygons (and polytopes)

A **lattice point** in $\mathbb{R}^n$ is a point with integer coordinates, i.e., an element of $\mathbb{Z}^n$.

A **(filled) convex lattice polytope** in $\mathbb{R}^n$ is a region which is the convex hull of finitely many lattice points. With a bit more effort one can also consider **nonconvex** lattice polytopes, but to simplify I'll skip this.

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The Ehrhart[‡] polynomial of a lattice polytope

For P a polytope in $\mathbb{R}^n$ and m a positive real number, define the dilation

$$mP = \{mx : x \in P\}.$$

Theorem (Ehrhart)

Let P be a convex lattice polytope in $\mathbb{R}^n$. Then there exists a polynomial $L_P(t) \in \mathbb{Q}[t]$ with the property that for each nonnegative integer m, $L_p(m)$ equals the number of interior and boundary lattice points in mP. In particular, $L_P(0) = 1$.

Assuming that P has positive volume in $\mathbb{R}^n$ (i.e., it is not contained in a lower-dimensional affine space), $L_P(t)$ has degree n and its leading coefficient is the volume of P.

[‡] Eugène Ehrhart worked as a high school teacher in France and engaged in research mathematics in his free time. He published a series of articles about lattice polytopes in the mid-1960s. Only later did he receive his PhD thesis, at the age of 60!

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The Ehrhart reciprocity law

Theorem (Ehrhart)

Let P be a convex lattice polytope in $\mathbb{R}^n$ with positive volume. Then for the same polynomial $L_p(t)$, for each positive integer m, $(-1)^n L_p(-m)$ equals the number of interior only *lattice points in mP.*

This "lifts" to a deeper statement in algebraic geometry (Serre duality for toric varieties). By the same token, many other assertions in this subject (e.g., the formula of Pommersheim for tetrahedra) double as statements of elementary number theory and deeper facts in algebraic geometry.

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Pick's theorem and Ehrhart polynomials

Let's look closely at the case $n = 2$. For P a convex lattice polygon with area V,

$$L_P(t) = Vt^2 + at + 1$$

for some rational number a. Plugging in $t = 1, t = -1$ yields

$$L_P(1) = V + a + 1 = I + B$$
$$L_p(-1) = V - a + 1 = I.$$

Eliminating a recovers Pick's theorem:

$$V = I + \tfrac{1}{2}B - 1.$$

If we instead solve for a, we find that

$$a = V + 1 - I = \tfrac{1}{2}B.$$

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Coefficients of the Ehrhart polynomial

By the previous slide, we have geometric interpretations of all coefficients of the Ehrhart polynomial when $n = 2$. For $n \geq 3$, things are more complicated!

For P a convex lattice polytope in $\mathbb{R}^n$ with positive volume V,

$$L_P(t) = Vt^n + Bt^{n-1} + \cdots + 1$$

where B is half the sum of the volumes[§] of the $(n-1)$-dimensional faces of P. But the terms $\cdots$ are more mysterious.

[§] These volumes have to be normalized suitably. For example, when $n = 2$, the normalized volume of an edge is one less than the number of lattice points on that edge.

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Ehrhart polynomial coefficients for some tetrahedra

Pommersheim (1993) computed the Ehrhart polynomial of the tetrahedron with vertices $(0,0,0),(a,0,0),(0,b,0),(0,0,c)$. The coefficient of t is

$$\frac{1}{12}\left(\frac{ac}{b}+\frac{bc}{a}+\frac{ab}{c}+\frac{(ABC)^2}{abc}\right)+\frac{1}{4}(a+b+c+A+B+C)$$
$$-As\left(\frac{bc}{ABC},\frac{a}{BC}\right)-Bs\left(\frac{ac}{ABC},\frac{b}{AC}\right)-Cs\left(\frac{ab}{ABC},\frac{c}{AB}\right)$$

where $A=\gcd(b,c), B=\gcd(c,a), C=\gcd(a,b)$ and $s(p,q)$ is a **Dedekind sum**:

$$s(p,q)=\sum_{i=1}^{q}(\!(\tfrac{i}{q})\!)(\!(\tfrac{pi}{q})\!), \qquad (\!(x)\!)=\begin{cases} x-\lfloor x\rfloor-\frac{1}{2} & (x\notin\mathbb{Z}) \\ 0 & (x\in\mathbb{Z}). \end{cases}$$

Such sums first appeared in the theory of **modular forms**. An alternate approach to this and other Ehrhart polynomials was introduced by Diaz–Robins.

Contents

Minkowski sums and mixed volumes

Let $P_1, \dots, P_k$ be convex regions in $\mathbb{R}^n$ (not necessarily polytopes). We may now define the dilate $\lambda P_i = \{\lambda x : x \in P_i\}$ for any $\lambda \geq 0$, not necessarily an integer.

The **Minkowski sum** of $P_1, \dots, P_k$ is the convex polytope

$$P_1 + \cdots + P_k = \{x_1 + \cdots + x_k : x_1 \in P_1, \dots, x_k \in P_k\}.$$

Now assume $k = n$. For $\lambda_1, \dots, \lambda_n \geq 0$, the volume of $\lambda_1 P_1 + \cdots + \lambda_n P_n$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_n$; the coefficient of $\lambda_1 \cdots \lambda_n$ in this polynomial is called the **mixed volume** $V(P_1, \dots, P_n)$.

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Quermassintegrals and intrinsic volumes

Let P be a convex region in $\mathbb{R}^n$. Let B_n be the unit ball in $\mathbb{R}^n$. Then for all $t \geq 0$,

$$V(tP + B_n) = \sum_{j=0}^{n} \binom{n}{j} W_{n-j}(P) t^j$$

where $W_j(P)$ is the mixed volume of P (taken $n - j$ times) and B_n (taken j times). It is called the j-th **quermassintegral** of P.

Another normalization you might find in the literature: the j-th **intrinsic volume** of P is

$$V_j(P) = \binom{n}{j} \frac{W_{n-j}(P)}{V(B_{n-j})}.$$

By taking $t \to \infty$, we see that $W_n(P) = V_n(P) = V(P)$ is the usual volume. Meanwhile, $V_{n-1}(P)$ is half the surface area of P. Hmm...

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So what is really going on?

The fundamental question in this project is: what on earth is going on with this analogy??? And does it suggest any geometric interpretation of the mysterious Ehrhart polynomial coefficients?

Possibly related question: is there a sensible simultaneous generalization of these two concepts? After all, if I go back to the equation

$$V(tP + B_n) = \sum_{j=0}^{n} \binom{n}{j} W_{n-j}(P) t^j$$

and specialize P to be a lattice polytope, then I can imagine replacing the ball B_n with a point and replacing the usual (Lebesgue) measure on $\mathbb{R}^n$ with a discrete measure concentrated on $\mathbb{Z}^n$; this then looks a lot like counting lattice points.

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Mixed Ehrhart polynomials

One clue might be the definition of **mixed Ehrhart polynomials** (Haase–Juhnke-Kubitzke–Sanyal–Theobald). For $P_1, \dots, P_k$ convex lattice polytopes in $\mathbb{R}^n$, the mixed Ehrhart polynomial $L_{P_1,\dots,P_k}(t)$ has the property that for each nonnegative integer m,

$$L_{P_1,\dots,P_k}(m) = \sum_{J \subseteq \{1,\dots,k\}} (-1)^{k-\#J} \#(\mathbb{Z}^n \cap \sum_{j \in J} mP_j)$$

(where $\sum_{j \in J} mP_j = 0$ when $J = \emptyset$). These are related to **discrete mixed volumes** introduced by Bihan.

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Positivity properties

Another way to milk this analogy is to compare positivity statements between the discrete and continuous settings. For example, mixed volumes are subject to the **Alexandrov–Fenchel inequality**

$$V(P_1, \dots, P_n) \geq \sqrt{V(P_1, P_1, P_3, \dots, P_n) V(P_2, P_2, P_3, \dots, P_n)}.$$

Does this have a discrete analogue?

Intrinsic nature of these coefficients

A theorem of Hadwiger asserts that the only "natural" measures of convex bodies are linear combinations of the intrinsic volumes. Is there a similar characterization of the Ehrhart polynomial coefficients? For instance, they are invariants for scissors congruence; are they the only such invariants? (This is true for $n = 2$. What about $n = 3$?)