Semistable reduction for overconvergent F-isocrystals: geometric aspects of the proof

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All of my papers referenced (and these slides) can be found at

`http://math.mit.edu/~kedlaya/papers/`

References

References to I, II, III, IV refer to the papers in the series "Semistable reduction for overconvergent F-isocrystals".

I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212
II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672
III: Local semistable reduction at monomial valuations, *Compos. Math.* **145** (2009), 143-172
IV: Local semistable reduction at nonmonomial valuations, arXiv 0712.3400v3 (2009); submitted

For more information and additional references, see `http://math.mit.edu/~kedlaya/papers/semistable.shtml`.

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Categories of isocrystals

Throughout, let k be a field of characteristic $p > 0$. Let $X \subseteq Y$ be an open dense immersion of k-varieties. We consider the following categories:

$(F\text{-})\mathrm{Isoc}^\dagger(X,Y)$: ($F$-)isocrystals on X overconvergent within Y
$(F\text{-})\mathrm{Isoc}(X)$: $=(F\text{-})\mathrm{Isoc}^\dagger(X,X)$ (*convergent isocrystals*)
$(F\text{-})\mathrm{Isoc}^\dagger(X)$: $=(F\text{-})\mathrm{Isoc}^\dagger(X,Y)$ with Y proper (*overconvergent isocrystals*)

For X smooth, elements of $\mathrm{Isoc}^\dagger(X,Y)$ are (locally) defined as vector bundles with integrable connection on a certain rigid analytic subspace of the (Raynaud) generic fibre of a p-adic formal lift of Y (a strict neighborhood of the tube of X), plus a convergence condition on the Taylor isomorphism.

These occur as analogues in rigid cohomology of smooth (lisse) ℓ-adic étale sheaves. The *unit-root* isocrystals correspond to p-adic representations of étale fundamental groups (Katz, Crew, Tsuzuki; see KSK, Swan conductors for p-adic differential modules II), but general isocrystals do not.

A rigidity property

We will use frequently the following rigidity property. (Can one drop smoothness using cohomological descent of Chiarellotto-Tsuzuki?)

Theorem (I, 5.2.1; II, 4.2.1; see also Caro, arXiv:0905.2210)

Suppose X is smooth and $U \subseteq X$ is open dense. The restriction functors

$$(F\text{-})\operatorname{Isoc}^\dagger(X,Y) \to (F\text{-})\operatorname{Isoc}^\dagger(U,Y)$$
$$F\text{-}\operatorname{Isoc}^\dagger(X,Y) \to F\text{-}\operatorname{Isoc}(X)$$

are fully faithful.

Logarithmic isocrystals

By a *smooth pair*, we mean a pair (X, Z) with X smooth over k and Z a strict normal crossings divisor on X. Consider categories:

$(F\text{-})\mathrm{Isoc}^{\log}(X,Z)$: convergent log-$(F$-$)$isocrystals on (X,Z) (Shiho)
$(F\text{-})\mathrm{Isoc}^{\mathrm{nil}}(X,Z)$: convergent log-$(F$-$)$isocrystals on (X,Z) with nilpotent residues along Z (see next slide)

Warning: it is *not known* how to construct a category of overconvergent log-$(F$-$)$isocrystals on (X,Z). We instead compute using local models, without assuming that these are independent of the choice of lifts. (This depends on rigidity.)

Logarithmic isocrystals with nilpotent residues

Let (X,Z) be a smooth pair, let D be a component of Z, and put $Z' = Z \setminus D$. We may restrict $\mathscr{E} \in \mathrm{Isoc}^{\log}(X,Z)$ to $(\mathscr{E}_D, N_D)$ where $\mathscr{E}_D \in \mathrm{Isoc}^{\log}(D, Z' \cap D)$ and $N_D \in \mathrm{Hom}(\mathscr{E}_D, \mathscr{E}_D)$ is horizontal. Call N_D the *residue* of $\mathscr{E}$ along D. By definition, $\mathscr{E} \in \mathrm{Isoc}^{\mathrm{nil}}(X,Z)$ iff N_D is nilpotent for all D. This is automatic if $\mathscr{E}$ carries a Frobenius, i.e.,

$$F\text{-}\mathrm{Isoc}^{\log}(X,Z) = F\text{-}\mathrm{Isoc}^{\mathrm{nil}}(X,Z)$$

(but not if only $\mathscr{E}|_U$ carries a Frobenius for some open dense $U \subseteq X$).

Theorem (I, 6.4.5)

The restriction functor

$$(F\text{-})\,\mathrm{Isoc}^{\mathrm{nil}}(X,Z) \to (F\text{-})\,\mathrm{Isoc}^{\dagger}(X \setminus Z, X)$$

is fully faithful. (This fails without requiring nilpotent residues.)

Alterations

An *alteration* $f: Y' \to Y$ is a proper, dominant, generically finite morphism. If k is perfect, we also assume f is generically étale.

Theorem (de Jong)

Let $X \subseteq Y$ be an open dense immersion of k-varieties. Then there exists an alteration $f: Y' \to Y$ such that $(Y', f^{-1}(Y \setminus X))$ is a smooth pair.

It is *not known* whether f can be taken to be finite over the regular locus of Y.

de Jong's theorem stands in for resolution of singularities over k. However, knowing resolution would not improve our main theorem *except* possibly by eliminating blowups outside the regular locus.

The semistable reduction theorem

Theorem (Semistable reduction; conjectured by Shiho)

Let $X \subseteq Y$ be an open immersion of k-varieties. For $f : Y' \to Y$ an alteration, put $X' = f^{-1}(X)$ and $Z' = Y' \setminus X'$. Then for any $\mathscr{E} \in F\text{-Isoc}^{\dagger}(X,Y)$, we can choose f so that (Y', Z') is a smooth pair and $f^*\mathscr{E}$ is the restriction (uniquely) of an element of $F\text{-Isoc}^{\text{nil}}(Y', Z')$.

Semistable reduction is used by Caro and Tsuzuki to prove overholonomicity of overconvergent F-isocrystals, and by Shiho to construct generic higher direct images in relative rigid cohomology.

An analogue in characteristic 0: a higher-dimensional version of Turrittin's structure theorem for formal connections. See KSK, Good formal structures for flat meromorphic connections I, II.

(Notes for experts: I assume *discretely valued* coefficients, and I don't know anything about what happens without Frobenius structure.)

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Local monodromy for isocrystals

Let (X,Z) be a smooth pair, put $U = X \setminus Z$, and let D be a component of Z with generic point η. For $\mathscr{E} \in (F\text{-})\,\mathrm{Isoc}^\dagger(U,X)$, the *local monodromy module* of $\mathscr{E}$ is the restriction $\mathscr{E}_D$ of $\mathscr{E}$ to

$$(F\text{-})\,\mathrm{Isoc}^\dagger(\mathrm{Spec}\,\mathrm{Frac}(\widehat{\mathscr{O}_{X,\eta}}), \mathrm{Spec}\,\widehat{\mathscr{O}_{X,\eta}}),$$

given an appropriate definition of this category. (Concretely, these are finite free modules with connection over a ring of convergent Laurent series, i.e., a bounded Robba ring.)

Zariski-Nagata purity

We have analogues of (the easy case of) Zariski-Nagata purity.

Theorem (I, 5.2.1; extended by Shiho, arXiv:0806.4394)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. Then the essential image of $\mathrm{Isoc}^\dagger(X,Y) \to \mathrm{Isoc}^\dagger(U,Y)$ consists of those $\mathscr{E}$ for which $\mathscr{E}_D$ extends to $\mathrm{Isoc}(\mathrm{Spec}\,\widehat{\mathscr{O}_{X,\eta}})$ for each codimension 1 component D of $X \setminus U$.

Theorem (I, 6.4.5; extended by Shiho, arXiv:0806.4394)

Let (X,Z) be a smooth pair, and put $U = X \setminus Z$. Then the essential image of $\mathrm{Isoc}^{\mathrm{nil}}(X,Z) \to \mathrm{Isoc}^\dagger(U,X)$ consists of those $\mathscr{E}$ for which $\mathscr{E}_D$ extends to $\mathrm{Isoc}^{\mathrm{nil}}(\mathrm{Spec}\,\widehat{\mathscr{O}_{X,\eta}},\eta)$ for each component D of Z.

In these cases, I'll say $\mathscr{E}_D$ is *extendable* (resp. *log-extendable*).

Some sample corollaries

Theorem (I, 5.3.1)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. For $\mathscr{E} \in \mathrm{Isoc}^\dagger(X,Y)$, any subobject of $\mathscr{E}$ in $\mathrm{Isoc}^\dagger(U,Y)$ lifts to a subobject of $\mathscr{E}$ in $\mathrm{Isoc}^\dagger(X,Y)$.

This follows because the property that $\mathscr{E}_D$ is constant passes to all subobjects. (Beware: the analogous statement for the restriction $\mathrm{Isoc}^\dagger(X) \to \mathrm{Isoc}(X)$ is false! Consider, e.g., a unit-root subcrystal.)

Theorem (I, 5.3.7)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. Then

$$\mathrm{Isoc}(X) \times_{\mathrm{Isoc}^\dagger(U,X)} \mathrm{Isoc}^\dagger(U,Y) = \mathrm{Isoc}^\dagger(X,Y).$$

(With Frobenius structures, one can multiply over F-$\mathrm{Isoc}(U)$.)

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Krull valuations

Let X be an irreducible k-variety. A *Krull valuation* on $k(X)$ over k is a function $v : k(X) \to \Gamma \cup \{\infty\}$ for some totally ordered group Γ, such that:

- $v(x) = \infty$ iff $x = 0$, and $v(x) = 0$ for all $x \in k^\times$;
- $v(xy) = v(x) + v(y)$;
- $v(x+y) \geq \min\{v(x), v(y)\}$.

Define

$$\begin{aligned}
\Gamma_v &= v(k(X)^\times) \qquad \textit{(value group)} \\
\mathfrak{o}_v &= \{x \in k(X) : v(x) \geq 0\} \qquad \textit{(valuation ring)} \\
\mathfrak{m}_v &= \{x \in k(X) : v(x) > 0\} \qquad \textit{(maximal ideal)} \\
\kappa_v &= \mathfrak{o}_v / \mathfrak{m}_v \qquad \textit{(residue field)}
\end{aligned}$$

The *center* of v on X is $\{x \in X : \mathfrak{o}_{X,x} \subseteq \mathfrak{o}_v\}$. If nonempty (e.g., if X is proper), it is closed and irreducible of dimension $\leq \operatorname{trdeg}(\kappa_v/k)$, and v is *centered on X*.

Divisorial valuations and semistable reduction

We say v is *divisorial* if v measures order of vanishing along some divisor on some variety birational to X. In particular, $\Gamma_v \cong \mathbb{Z}$.

Let $X \subseteq Y$ be an open immersion of irreducible k-varieties. For $\mathscr{E} \in \mathrm{Isoc}^\dagger(X, Y)$, we get a local monodromy module $\mathscr{E}_v$ for each divisorial valuation v on $k(X)$ centered on Y.

Theorem (approximately II, 3.4.4)

$\mathscr{E}$ admits semistable reduction if and only if there exists a finite cover $X' \to X$ with X' irreducible, such that for each divisorial valuation v on $k(X)$ centered on Y, for some extension w of v to $k(X')$, $\mathscr{E}_w$ is log-extendable.

One can achieve this for a *single* v using the theorem of André-Mebkhout-KSK (Crew's conjecture). However, that plus Zariski-Nagata purity do not suffice: we cannot control singularities of X'.

Riemann-Zariski spaces

Let $S_{k(X)/k}$ be the set of equivalence classes of Krull valuations on $k(X)$ over k. (Here $v \sim v'$ iff $\mathfrak{o}_v = \mathfrak{o}_{v'}$.) This carries the *Zariski-Hausdorff topology*, specified by the basis of opens given by

$$\{v \in S_{k(X)/k} : v(f_1), \dots, v(f_m) \geq 0; v(g_1), \dots, v(g_n) > 0\}$$

for any $f_1, \dots, f_m, g_1, \dots, g_n \in k(X)$.

Theorem (Zariski)

The topological space $S_{k(X)/k}$ is compact.

Local semistable reduction

Let $X \subseteq Y$ be an open immersion of irreducible k-varieties. For $f : Y' \to Y$ an alteration, put $X' = f^{-1}(X)$ and $Z' = Y' \setminus X'$.

For $\mathscr{E} \in F\text{-Isoc}^\dagger(X,Y)$ and $v \in S_{k(X)/k}$ centered on Y, $\mathscr{E}$ admits *local semistable reduction at v* if there exists an alteration $f : Y' \to Y$ with Y' irreducible and an open $U \subseteq Y'$ such that $(U, U \cap Z')$ is a smooth pair, some extension of v to $k(Y')$ is centered on U, and $f^*\mathscr{E}$ lifts from $F\text{-Isoc}^\dagger(X' \cap U, U)$ to $F\text{-Isoc}^{\mathrm{nil}}(U, U \cap Z')$.

Using Zariski's compactness theorem, we obtain the following.

Theorem (II, 3.3.4, 3.4.5; IV, 2.4.2)

Suppose that $\mathscr{E}$ admits local semistable reduction at all $v \in S_{k(X)/k}$ centered on Y, Then $\mathscr{E}$ admits semistable reduction.

Note: we need all v, not just divisorial valuations, so $\mathscr{E}_v$ may not make sense.

Abhyankar's inequality

The *height* (*real rank*) of v is the minimum m such that Γ_v embeds into the lexicographic product $\mathbb{R}^m$.

The *rational rank* of v is $\dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$. Note that $\mathrm{height}(v) \leq \mathrm{ratrank}(v)$.

The *transcendence defect* of v is

$$\mathrm{trdefect}(v) = \dim(X) - \mathrm{ratrank}(v) - \mathrm{trdeg}(\kappa_v/k).$$

Theorem (Abhyankar)

For any $v \in S_{k(X)/k}$, $\mathrm{trdefect}(v) \geq 0$. Moreover, if $\mathrm{trdefect}(v) = 0$, then $\Gamma_v \cong \mathbb{Z}^{\mathrm{ratrank}(v)}$ and κ_v is finitely generated over k.

A valuation v with $\mathrm{trdefect}(v) = 0$ is called an *Abhyankar valuation*. These are dense in $S_{k(X)/k}$, since they include divisorial valuations.

Reductions

Theorem (II, 3.2.6)

To prove (local) semistable reduction for a given isocrystal, it suffices to do so after base change from k to k^{alg}.

Theorem (II, 4.2.4, 4.3.4)

To prove local semistable reduction over an algebraically closed field k, it suffices to do so for all valuations v with $\operatorname{height}(v) = 1$ and $\kappa_v = k$.

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Local uniformization for Abhyankar valuations

Assume from now on that $k = k^{\mathrm{alg}}$. Let $X \subseteq Y$ be an open dense immersion of irreducible k-varieties. Let v be a valuation on $k(X)$ over k centered on Y with $\mathrm{trdefect}(v) = 0$ and $\kappa_v = k$.

Theorem (Kuhlmann, Knaf)

There is a blowup Y' of Y and local coordinates $t_1, \dots, t_n$ on Y' at the center of v, such that

$$\alpha_1 = v(t_1), \dots, \alpha_n = v(t_n)$$

are linearly independent over $\mathbb{Q}$ and generate Γ_v as a $\mathbb{Z}$-module.

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, let v_β denote the $(\beta_1, \dots, \beta_n)$-Gauss valuation in terms of $t_1, \dots, t_n$. Then the completion $k(X)_v$ is isomorphic to the v_α-completion of $k[t_1^{\pm}, \dots, t_n^{\pm}]$.

Differential ramification breaks

Take $\mathscr{E} \in \mathrm{Isoc}^\dagger(X,Y)$ of rank d. Fix $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$. We may realize $\mathscr{E}$ as a ∇-module on a subspace of the rigid or nonarchimedean analytic $(t_1, \dots, t_n)$-affine space including (for some $\varepsilon \in (0,1)$)

$$\{(t_1, \dots, t_n) : (|t_1|, \dots, |t_n|) = (\rho^{\beta_1}, \dots, \rho^{\beta_n}) \text{ for some } \rho \in (\varepsilon, 1)\}.$$

Then there exist $b_1(\mathscr{E}, \beta) \geq \cdots \geq b_d(\mathscr{E}, \beta) \geq 0$ such that the intrinsic subsidiary generic radii of convergence at $(|t_1|, \dots, |t_n|) = (\rho^{\beta_1}, \dots, \rho^{\beta_n})$ are equal to $\rho^{b_1(\mathscr{E},\beta)}, \dots, \rho^{b_d(\mathscr{E},\beta)}$. These are the *differential ramification breaks* of $\mathscr{E}$ along v_β (at least if $\beta \in \mathbb{Q}^n$).

In the one-dimensional case, these are ordinary ramification breaks of the local monodromy representation (Crew, Matsuda, Tsuzuki). For more discussion, see: KSK, Swan conductors for p-adic differential modules, I, II.

Variation of differential Swan conductors

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Q}^n$, the differential ramification breaks satisfy

$$d! b_i(\mathscr{E}, \beta) \in \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_d \qquad (i = 1, \dots, d).$$

Moreover, $b_1(\mathscr{E}, \beta) = 0$ if and only if $\mathscr{E}_{v_\beta}$ becomes unipotent after pulling back along a cover tamely ramified along $t_1 \cdots t_n = 0$ (III, 5.2.5). Define

$$B_i(\mathscr{E}, \beta) = b_1(\mathscr{E}, \beta) + \cdots + b_i(\mathscr{E}, \beta).$$

Theorem (III, 2.4.2, 4.4.7 for $i = 1$; KSK, Xiao in general)

The functions $d! B_i(\mathscr{E}, \beta)$ and $B_d(\mathscr{E}, \beta)$ are convex and piecewise integral affine (integral polyhedral) on $\beta \in [0, +\infty)^n$.

Two approaches to local semistable reduction

Original approach (III, 6.3.1): use an analogue of the p-adic local monodromy theorem (KSK, The p-adic local monodromy theorem for fake annuli) to reach a situation (after suitable alteration) where $b_1(\mathscr{E},\alpha) = 0$. Since $d!b_1(\mathscr{E},\beta) = d!B_1(\mathscr{E},\beta)$ is integral polyhedral, this forces $b_1(\mathscr{E},\beta)$ to vanish identically in a neighborhood of α.

Alternate approach (sketched in IV, appendix): imitate Mebkhout's proof of the monodromy theorem, replacing Christol-Mebkhout decomposition theory with its higher-dimensional analogue (KSK-Xiao).

Both of these depend crucially on being able to describe v in local coordinates. For the case $\mathrm{trdefect}(v) > 0$, a new idea is needed.

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Setup

Again, assume that v is a valuation on $k(X)$ centered on v with $\kappa_v = k$, but now suppose $\mathrm{trdefect}(v) = m > 0$. Assume local semistable reduction for all w with $\mathrm{trdefect}(w) < m$.

Unfortunately, v does not admit a sufficiently convenient description in local coordinates to permit an analogue of our argument in the transcendence defect 0 case. This is in part because Γ_v need not be finitely generated over $\mathbb{Z}$.

Instead, argue by induction on transcendence defect. Related arguments:

- Temkin, Inseparable local uniformization, arXiv:0804.1554.
- KSK, Good formal structures for formal meromorphic connections II.
- Possibly an application to multiplier ideals, following Boucksom, Favre, Jonsson.

Induction on transcendence defect via fibrations

Choose a fibration $\pi : Y \to Y^0$ in curves such that $k(Y^0)$ contains a $\mathbb{Q}$-basis of $\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the restriction v^0 of v to $k(Y^0)$ satisfies

$$\mathrm{trdefect}(v^0) = \mathrm{trdefect}(v) - 1 = m - 1.$$

Let $z \in Y$ be the center of v, and put $z^0 = \pi(z)$. Let $x \in k(Y)$ restrict to $\pi^{-1}(z^0)$ giving a local parameter at z.

We allow the following operations on the geometric data (IV, 5.1.6–9).

(i) Change base: replace Y^0 by an alteration or a finite radicial cover.

(ii) Blow up: adjoin $(x-g)/h$ for $g, h \in k(Y^0)$ with $v(x-g) > v(h) \geq 0$.

(iii) Make a tame cover: adjoin $x^{1/m}$ for some positive integer m coprime to p.

(iv) Make a *special Artin-Schreier cover*: adjoin uf such that $u^p - u = y/f^p$ with $f \in k(Y^0)$ and $y \in \lambda x + \mathfrak{m}_{X,z}$ for some $\lambda \in k^\times$.

A path in valuation space

We identify v with a multiplicative seminorm on $k(Y^0)_{v^0}[\![x]\!]$ bounded by the 1-Gauss norm, corresponding to a point of type 1 (classical) or 4 (spherical) in the Berkovich open unit disc $\mathbb{D}$ over $k(Y^0)_{v^0}$.

Using pointwise comparison on $k(Y^0)_{v^0}[\![x]\!]$ as a partial ordering, $\mathbb{D}$ forms a tree in which v is a leaf. Let $\mathscr{P}$ be the branch ending at v. Each $w \in \mathscr{P} \setminus \{v\}$ "is" a valuation on $k(Y)$ with transcendence defect $m-1$, so $\mathscr{E}$ admits local semistable reduction at w.

Note that we can shorten $\mathscr{P}$ (on the side away from v) using the blowup operation (ii) on the geometric data.

Local monodromy representations

For each valuation w at which local semistable reduction is known, one obtains a (semisimplified) local monodromy representation τ_w of the inertia subgroup I_w of $\pi_1^{\mathrm{et}}(k(Y),*)$; it is a linear representation having *finite image* (IV, 2.5.2). If τ_w is trivial and (Y,Z) is a smooth pair, then $\mathscr{E}$ is log-extendable on some open dense subscheme of Y on which w is centered.

This applies to $w \in \mathscr{P} \setminus \{v\}$. The result is reflected in the tensor category of $\mathscr{E}$ restricted to some annulus centered at v. In particular, if this category is trivial, then we can kill the local monodromy representation at w using a base change. We then get local semistable reduction at v via Zariski-Nagata purity.

Strategy: reduce to this case by showing that the tensor category structures for different w's are "coherent", finding a subobject corresponding to a special Artin-Schreier character, trivializing this character, and repeating. Only finitely many iterations possible, after which we win. (Compare Mebkhout's proof of Crew's conjecture.)

Numerical invariants

Identify $\mathscr{P}$ with an interval $(0, s_0]$ by identifying w with $s = -\log \operatorname{radius}(w)$. (The *radius* of $w \in \mathscr{P}$ is the infimum of the radii of discs containing w.)

We define certain numerical invariants $b_1(\mathscr{E}, s) \geq \cdots \geq b_d(\mathscr{E}, s) \geq s$ for $w \in \mathscr{P}$, akin to the differential ramification breaks (IV, 3.1.3, 5.2.3). Put $B_i(\mathscr{E}, s) = b_1(\mathscr{E}, s) + \cdots + b_i(\mathscr{E}, s)$.

Theorem (IV, 3.1.4, 4.7.5)

The $d!B_i(\mathscr{E}, s)$ are convex and piecewise affine with integral slopes; the slopes are nonpositive as long as $b_i(\mathscr{E}, s) > s$. Moreover, in a neighborhood of v, each $b_i(\mathscr{E}, s)$ is either constant or identically s.

This uses quantitative Christol-Mebkhout theory (KSK, p-adic differential equations) plus some difficult calculations. (Reduces to: for $Q \in k(Y)[T]$, the Newton polygon of Q at a point of $\mathscr{P}$ becomes constant near v.)

Endgame

Using what we have so far, we can force the situation where $\mathscr{E}$ and $\mathscr{E}^\vee \otimes \mathscr{E}$ decompose the same way over any annulus centered at v. Unless we are in the good case, we can force one of these to have a rank 1 subquotient $\mathscr{F}$ whose p^n-th tensor power is trivial for some $n > 0$. (Reduces to: for τ a nontrivial linear representation of a finite p-group whose coefficient field is algebraically closed of characteristic 0, either τ or $\tau^\vee \otimes \tau$ has a nontrivial rank 1 subrepresentation.)

Using some careful analysis of Dwork isocrystals (and again quantitative Christol-Mebkhout), we show that $\mathscr{F}^{\otimes p^{n-1}}$ is trivialized by a *special* Artin-Schreier cover (II, 5.6.3).

This can only happen finitely many times, after which we win.