Sato-Tate groups of abelian surfaces and threefolds

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FKRS: Sato-Tate distributions and Galois endomorphism modules in genus 2, *Compositio Mathematica* **148** (2012), 1390–1442.
FKS: Sato-Tate groups of some weight 3 motives, arXiv:1212.0256v2 (2013).
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Elliptic curves

Let E be an elliptic curve over a number field K. For $\mathfrak{p}$ a prime ideal of K at which E has good reduction, let $a_\mathfrak{p}$ be the trace of Frobenius of E at $\mathfrak{p}$, so that

$$\#E(\mathfrak{o}_K/\mathfrak{p}) = q + 1 - a_\mathfrak{p} \qquad (q = \mathrm{Norm}(\mathfrak{p})).$$

By a theorem of Hasse, we have $|a_\mathfrak{p}| \leq 2\sqrt{q}$; that is, the normalized characteristic polynomial of Frobenius splits over $\mathbb{C}$ as

$$\overline{L}_\mathfrak{p}(T) = T^2 - \frac{a_\mathfrak{p}}{\sqrt{q}} T + 1 = (T - \alpha)(T - \beta)$$

with $|\alpha| = |\beta| = 1$.

Equidistribution: the Sato-Tate conjecture

Define a compact Lie group G as follows.

- If E has complex multiplication defined over K, put $G = \mathrm{SO}(2)$.
- If E has complex multiplication not defined over K, let G be the normalizer of SO(2) in SU(2); it consists of two connected components, one of which is SO(2).
- If E does not have complex multiplication, put $G = \mathrm{SU}(2)$.

Conjecture (Sato-Tate)

As $\mathfrak{p}$ varies, the $\overline{L}_{\mathfrak{p}}(T)$ become equidistributed for the image of the Haar measure of G via the characteristic polynomial map.

Theorem (Taylor et al.)

This conjecture holds if E has complex multiplication or if K is totally real.

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The Sato-Tate group of an abelian variety

Let A be an abelian variety of dimension g over K. Fix an embedding $K \hookrightarrow \mathbb{C}$. Then singular homology $H_1(A_{\mathbb{C}}^{\mathrm{top}}, \mathbb{Q})$ carries a symplectic form (cup product), on which we have actions of:

- the unitary symplectic group $\mathrm{USp}(2g)$;
- the endomorphism algebra $\mathrm{End}(A_{\overline{K}})$.

Let $\mathrm{ST}(A)$ be the set of $\gamma \in \mathrm{USp}(2g)$ such that for some $\tau = \tau(\gamma) \in G_K$, we have $\gamma^{-1}\alpha\gamma = \tau(\alpha)$ for all $\alpha \in \mathrm{End}(A_{\overline{K}})$.

By construction, $\mathrm{ST}(A)$ is a compact (but often disconnected) Lie group, and the component group $\mathrm{ST}(A)/\mathrm{ST}(A)^\circ$ receives a continuous surjection from G_K.

Warning: this is not the correct definition in general! However, it does give the correct group for $g \leq 3$.

The equidistribution conjecture

For each prime ideal $\mathfrak{p}$ of K at which A has good reduction, it is possible to associate to $\mathfrak{p}$ a conjugacy class $g_{\mathfrak{p}}$ in $\mathrm{ST}(A)$. (Sketch: take the action on the ℓ-adic Tate module for some prime ℓ, choose a field inclusion $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, then rescale.)

Conjecture (Sato-Tate for abelian varieties)

As $\mathfrak{p}$ varies, the classes $g_{\mathfrak{p}}$ become equidistributed for the image of the Haar measure on $\mathrm{ST}(A)$ (for the correct definition of $\mathrm{ST}(A)$).

This is backed by numerical evidence for $g \leq 3$:

`http://math.mit.edu/~drew`.

Theorem (Johansson, 2013, following Hecke)

The Sato-Tate conjecture for abelian varieties is true whenever $\mathrm{ST}(A)^0$ is a torus (in which case the above definition coincides with the correct one).

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Sato-Tate groups and endomorphism algebras

Theorem (FKRS, 2013)

The Sato-Tate group of an abelian surfaces determines, and is determined by, the noncommutative ring $\mathrm{End}_{\overline{K}}(A) \otimes_{\mathbb{Z}} \mathbb{R}$ *equipped with its* G_K*-action.*

Note that since we tensor with $\mathbb{R}$, the Sato-Tate group fails to distinguish some split and nonsplit cases, e.g., when $\mathrm{End}_{\overline{K}}(A)$ equals $\mathbb{Z} \oplus \mathbb{Z}$ versus $\mathbb{Z}(\sqrt{2})$.

A classification theorem for abelian surfaces

Theorem (FKRS, 2013)

Up to conjugation within $\mathrm{USp}(4)$*, there are exactly* 52 *possible Sato-Tate groups for abelian surfaces over number fields. Of these,* 34 *occur over* $\mathbb{Q}$ *and* 35 *occur over totally real fields.*

The most complicated part of the classification occurs when $\mathrm{ST}(A)^0$ is a one-dimensional torus: the component group can have up to 48 elements!

Both theorems are obtained using a classification argument on compact Lie groups (which yields 55 cases) plus matching to endomorphism algebras (which eliminates 3 cases). Achieving this classification depended heavily on numerical evidence, especially the computation of *moment statistics*.

Verification of the conjecture

We have empirically verified equidistribution for examples of all 52 Sato-Tate groups (again see `http://math.mit.edu/~drew`). On the theoretical side, the case of USp(4) seems intractable with present technology, but otherwise...

Theorem (Johansson, 2013)

The Sato-Tate conjecture for abelian surfaces is true when $\mathrm{ST}(A) \neq \mathrm{USp}(4)$ *and (a certain quadratic extension of) K is totally real.*

The main content is when $\mathrm{ST}(A)^0$ is $\mathrm{SU}(2)$ or $\mathrm{SU}(2) \times \mathrm{SU}(2)$; this uses potential automorphy (especially Patrikis–Taylor).

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What about abelian threefolds?

In principle, the same methodology as for abelian surfaces should apply to classify Sato-Tate groups of abelian threefolds.

In practice, this process needs to be automated, as there appear to be hundreds (thousands?) of possible groups. We currently lack good software for dealing with compact Lie groups which are neither connected nor finite.

Sample: the following can occur as orders of component groups:

$$96, 168, 216.$$

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Sato-Tate groups of motives

There is an analogous construction for motives of higher weight (abelian varieties being the case of weight 1) using absolute Hodge cycles. See: KSK and G. Banaszak, An algebraic Sato-Tate group and Sato-Tate conjecture.

This also yields the correct definition for abelian varieties of dimension $g \geq 4$. This suggests the possibility of numerically identifying examples of abelian fourfolds whose Mumford-Tate groups are not determined by endomorphisms, via Frobenius statistics. (These were first shown to exist by Mumford, but not in a very constructive way.)

Another classification result

Consider motives of weight 3 with Hodge numbers 1,1,1,1 (e.g., symmetric cube of an elliptic curve, or certain Calabi-Yau threefolds). In this case, the Sato-Tate group is again a subgroup of USp(4). This time, group theory yields 26 possible Sato-Tate groups, of which 25 have been confirmed empirically.

The 26th group appears not to occur, but in this case we have no arithmetic matching argument available to rule it out. Other techniques for ruling out groups would be of value.