The tame Belyĭ theorem in positive characteristic

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The UC San Diego campus sits on unceded ancestral land of the Kumeyaay Nation. The Kumeyaay people continue to have an important and thriving presence in the region: https://www.kumeyaay.info.

Riemann rigidity of three-point covers

Let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a finite morphism of Riemann surfaces branched only over $\{0, 1, \infty\}$. By a theorem of Riemann, f admits no infinitesimal deformations.

If we regard X as an algebraic curve, then f becomes a finite morphism of curves branched only over $\{0, 1, \infty\}$. The moduli space of such curves is of finite type and (by the above) of dimension 0, so it consists of finitely many isolated (stacky) points. The action of $\mathrm{Aut}(\mathbb{C})$ must therefore act through $\mathrm{Gal}(K/\mathbb{Q})$ for some number field K.

The weak Belyĭ theorem

Theorem (Belyĭ, 1980)

Let X be an algebraic curve over an algebraically closed field of characteristic 0. Then X admits a finite morphism to $\mathbb{P}^1$ branched only over $\{0, 1, \infty\}$ if and only if X admits a model over some number field.

In this statement, the "only if" statement is a consequence of Riemann rigidity. The content is the "if" statement, which can be proved in a somewhat stronger form...

A finite morphism f from a curve X to $\mathbb{P}^1$ branched only over $\{0, 1, \infty\}$ is called a **Belyĭ map** on X. More generally, for S a finite set of closed points on X, f is a **Belyĭ map** on (X, S) if it carries both S and its branch locus into $\{0, 1, \infty\}$.

The strong Belyĭ theorem

Theorem (Belyĭ, 1980)

Let (X, S) be a marked curve over a number field K. Then (X, S) admits a Belyĭ map defined over K (not just over some finite extension of K).

The strategy: choose a sequence of finite morphisms $(X, S) \to (X', S')$ of marked curves such that at each step the branch locus maps into S', and at the last step the target is $(\mathbb{P}^1_K, \{0, 1, \infty\})$.

We may start with any finite map $X \to \mathbb{P}^1_K$. By enlarging S we may force it to be Galois-stable, and then run the rest of the argument taking $K = \mathbb{Q}$.

The strong Belyĭ theorem over $\mathbb{Q}$

Suppose now that $K = \mathbb{Q}$, $X = \mathbb{P}^1_{\mathbb{Q}}$, and S contains exactly n nonrational points $\alpha_1, \dots, \alpha_n$. Apply the map $\mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ defined by

$$P(x) = (x - \alpha_1) \cdots (x - \alpha_n);$$

the only nonrational points in the new S are the nonrational images of zeroes of $P'(x)$, of which there are at most $n - 1$.

By induction, we reduce to the case where $S \subseteq \mathbb{P}^1(\mathbb{Q})$. To treat the case $S = \{0, 1, \frac{m}{m+n}, \infty\}$ with m, n coprime positive integers, use the map

$$x \mapsto \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n,$$

which carries the four branch points $0, 1, \frac{m}{m+n}, \infty$ to $0, 0, 1, \infty$ respectively.

Rigidity fails in positive characteristic

Theorem

Let X be a curve over **any** *field of characteristic $p > 0$. Then X admits a finite morphism to $\mathbb{P}^1$ branched only over $\{\infty\}$.*

By marking X as before, we may reduce to the case $X = \mathbb{P}^1$. We can then choose a **linearized polynomial**

$$P(x) = x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x$$

which maps S to $\{0\}$ and has branch locus $\{\infty\}$ (because $P'(x) = a_n$).

Now note that the map

$$x \mapsto x^p + x^{-1}$$

has branch locus $\{\infty\}$ and carries $\{0, \infty\}$ to $\{\infty\}$.

Aside: a higher-dimensional analogue

Theorem (K, 2005)

Let X be a generically smooth projective variety of dimension n over a field of characteristic $p > 0$. Then X admits a finite morphism to $\mathbb{P}^n$ branched only over one hyperplane.

This suggests that there is no hope to prove resolution of singularities in characteristic p by studying finite morphisms to simple targets.

Tame ramification

A finite morphism $f : X \to X'$ of curves is **tame** if for each point $x \in X$, the ramification degree of $\mathcal{O}_{X',f(x)} \to \mathcal{O}_{X,x}$ is coprime to p. By Grothendieck's theory of tame fundamental groups, these maps again satisfy rigidity.

Consequently, any curve over a field of characteristic p admitting a tame Belyĭ map admits a model over some finite extension of $\mathbb{F}_p$.

The weak tame Belyĭ theorem

Theorem (Saïdi, 1997)

Let (X, S) be a curve over an algebraically closed field of characteristic $p > 2$. Then (X, S) admits a tame Belyĭ map if and only if (X, S) admits a model over some finite extension of $\mathbb{F}_p$.

Suppose X is a curve over a finite field k of characteristic p. A generic finite morphism $X \to \mathbb{P}^1$ has only simple branch points, and so is tamely ramified because $p \neq 2$. We may thus assume $X = \mathbb{P}^1_k$.

The construction of Saïdi

Suppose now that $X = \mathbb{P}^1_k$ for some finite field k. The set S is contained in $\{0, \infty\} \cup \mathbb{F}_q^\times$ for some power q of p. The map

$$x \mapsto x^{q-1}$$

is tamely ramified with branch locus $\{0, \infty\}$ and carries S into $\{0, 1, \infty\}$.

This step does not depend on having $p \neq 2$, nor on proving the weak vs. the strong Belyĭ theorem.

The strong Belyĭ theorem

Theorem (KLW, 2020)

Let X be a curve over a finite field k of characteristic $p > 2$. Then X admits a tame Belyĭ map defined over k.

By Saïdi's construction, the only issue is to produce a finite tame morphism $X \to \mathbb{P}^1_k$; because k is finite, it is not enough to know that tameness is a Zariski open condition on the map.

Instead, we show that for $d \gg 0$, a **random** finite morphism $X \to \mathbb{P}^1_k$ of degree d is tame with positive probability. This is (mostly) a consequence of Poonen's Bertini theorem over finite fields.

Reminder: the difficulty with tame morphisms

Let k be an algebraically closed field of characteristic 2. The only obstruction to proving the weak tame Belyĭ theorem over k is to prove that every curve X over k admits a finite tame morphism to $\mathbb{P}^1_k$. A generic finite morphism to $\mathbb{P}^1_k$ has only simple ramification, but double points cause wild ramification in characteristic 2.

One might guess that X always admits a map to $\mathbb{P}^1_k$ with only triple ramification points. This is not known even in characteristic 0.

This question has arithmetic nature: if we allow k to be a general field of characteristic 2, then in some cases there is no finite tame map to $\mathbb{P}^1_k$. This was shown by Schröer for a generic curve*, and by KLW for a certain genus-1 curve over the perfect closure of $\mathbb{F}_2(t)$.

*Meaning the generic fiber of the universal family over M_g rigified with full level N structure for some odd $N \geq 3$. Here k is imperfect.

A result of Sugiyama–Yasuda

Theorem (Sugiyama–Yasuda, 2019)

Let k be an **algebraically closed** *field of characteristic 2. Let X be a curve over k. Then X admits a finite tame morphism to $\mathbb{P}^1_k$.*

The argument is based† on a mod-2 analogue of the **Schwarzian derivative**:

$$f(z) \mapsto \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

which vanishes precisely on Möbius transformations. More on this later.

†The link with the Schwarzian derivative was only noticed in hindsight by Yuichiro Hoshi.

The weak tame Belyĭ theorem

Theorem (Sugiyama–Yasuda, Anbar–Tutdere, 2019)

Let X be a curve over an algebraically closed field of characteristic 2. Then X admits a tame Belyĭ map if and only if X admits a model over some finite extension of $\mathbb{F}_2$.

A variation on Sugiyama–Yasuda

Theorem (KLW, 2020)

Let k be a **finite** *field of characteristic* 2*. Let X be a curve over k. Then X admits a finite tame morphism to $\mathbb{P}^1_k$.*

As noted above, this does not hold for any field of characteristic 2. The proof yields, for any given curve X over a perfect field k of characteristic 2, a computable (in principle) obstruction that vanishes if and only if X admits a finite tame morphism to $\mathbb{P}^1_k$.

The case of ordinary elliptic curves

Theorem (KLW, 2020)

Let k be a perfect field of characteristic 2. let X be the elliptic curve

$$y^2 + xy = x^3 + ax^2 + b \qquad (a, b \in k; b \neq 0).$$

Let $\varphi : k \to k$ be the absolute Frobenius map $x \mapsto x^2$.

(a) *If one of $a, b, a+b$ is in the image of $\varphi + 1$, then X* **does** *admits a finite tame morphism to $\mathbb{P}^1_k$.*

(b) *If $X(k)$ is torsion and none of $a, b, a+b$ is in the image of $\varphi + 1$, then X* **does not** *admit a finite tame morphism to $\mathbb{P}^1_k$.*

Using results of Ghioca and Rössler, one can exhibit examples of elliptic curves over the perfect closure of $\mathbb{F}_2(t)$ to which (b) applies.

The symbol map

Let X be a curve over a perfect field of characteristic 2. Define the group $\Gamma = \mathrm{PGL}_2(k(X)^4)$; it acts freely on $k(X) \setminus k(X)^2$ via Möbius transformations.

The mod-2 Schwarzian derivative gives rise to a **symbol map**[‡]

$$\mathrm{SY} : (k(X) \setminus k(X)^2) \times (k(X) \setminus k(X)^2) \to \frac{k(X)}{k(X)^2}$$

such that:

- SY is symmetric and Γ-invariant in each argument;
- $\mathrm{SY}(f, g) = 0$ if and only if f and g are Γ-equivalent;
- $\mathrm{SY}(f, g) + \mathrm{SY}(g, h) = \mathrm{SY}(f, h)$.

[‡]Our terminology, so that we can label the map SY in honor of Sugiyama–Yasuda.

Pseudotame vs. tame morphisms

Following Sugiyama–Yasuda, we say that $f \in k(X)$ is **pseudotame** if at each closed point $x \in X$, f is Γ-equivalent to a tame function[§] depending on x.

A tame morphism is pseudotame, but not conversely. However, following Sugiyama–Yasuda, we show that **existence** of a pseudotame function implies existence of a tame function.

[§] We freely identify nonzero elements of $k(X)$ with finite maps from X to $\mathbb{P}^1_k$.

Tame morphisms from sections of conic bundles

Using the symbol map, we obtain a family of conic bundles on X such that X admits a pseudotame function iff one of the conic bundles has a section.

When k is algebraically closed, every conic bundle on X admits a section by Tsen's theorem. This recovers the result of Sugiyama–Yasuda.

When k is finite, every conic bundle on X admits a section by class field theory, specifically the Albert–Brauer–Hasse–Noether exact sequence

$$0 \to \mathrm{Br}(k(X)) \to \bigoplus_{x \in X^\circ} \mathrm{Br}(k(X)_x) \to \mathbb{Z} \to 0.$$

This yields our claimed result, and hence the strong tame Belyĭ theorem...

In conclusion: the strong tame Belyĭ theorem

Theorem (KLW, 2020 plus previous)

Let X be a curve over a finite field k. Then X admits a tame Belyĭ map defined over k.

Question: what can one say about the minimum degree of such a map?