Semistable reduction for overconvergent F-isocrystals

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References

References to I, II, III, IV refer to the papers in the series "Semistable reduction for overconvergent F-isocrystals".

I: Unipotence and logarithmic extensions, *Compos. Math.* **143** (2007), 1164–1212
II: A valuation-theoretic approach, *Compos. Math.* **144** (2008), 657–672
III: Local semistable reduction at monomial valuations, arXiv math/0609645v3 (2008); to appear in *Comp. Math.*
IV: Local semistable reduction at nonmonomial valuations, arXiv 0712.3400v2 (2008); submitted

For more information and additional references, see `http://math.mit.edu/~kedlaya/papers/semistable.shtml`.

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Notations

k: a field of characteristic $p > 0$
K: a complete discretely valued field ($\mathrm{char}(K) = 0$) with residue field k
q: a power of p
σ_K: a continuous endomorphism of K lifting the q-power Frobenius on k

Warning: all isocrystals will be defined using the coefficient field K and the Frobenius lift σ_K, *without* including them in the notation.

Isocrystals

Let $X \subseteq Y$ be an open dense immersion of k-varieties. Consider categories:

(F-)$\mathrm{Isoc}^\dagger(X,Y)$: ($F$-)isocrystals on X overconvergent within Y
(F-)$\mathrm{Isoc}(X)$: =(F-)$\mathrm{Isoc}^\dagger(X,X)$ (*convergent F-isocrystals*)
(F-)$\mathrm{Isoc}^\dagger(X)$: =(F-)$\mathrm{Isoc}^\dagger(X,Y)$ with Y proper (*overconvergent F-isocrystals*)

Theorem (I, 5.2.1; II, 4.2.1)

Suppose X is smooth and $U \subseteq X$ is open dense. The restriction functors

$$(F\text{-})\,\mathrm{Isoc}^\dagger(X,Y) \to (F\text{-})\,\mathrm{Isoc}^\dagger(U,Y)$$
$$F\text{-}\mathrm{Isoc}^\dagger(X,Y) \to F\text{-}\mathrm{Isoc}(X)$$

are fully faithful.

The latter is conjectured to hold without F (Tsuzuki).

Logarithmic isocrystals

By a *smooth pair*, we mean a pair (X, Z) with X smooth over k and Z a strict normal crossings divisor on X. Consider categories:

(F-)$\mathrm{Isoc}((X,Z))$: convergent log-(F-)isocrystals on (X,Z) (Shiho)
(F-)$\mathrm{Isoc}^{\mathrm{nil}}((X,Z))$: convergent log-($F$-)isocrystals on (X,Z) with nilpotent residues along Z (see below)

Warning: it is *not known* how to construct $\mathrm{Isoc}^{\dagger}((X,Z))$. However, we will use local models of this category, without assuming that these are independent of the choice of lifts.

Logarithmic isocrystals (contd.)

Let (X,Z) be a smooth pair, let D be a component of Z, and put $Z' = Z \setminus D$. We may restrict $\mathscr{E} \in \mathrm{Isoc}((X,Z))$ to $(\mathscr{E}_D, N_D)$ where $\mathscr{E}_D \in \mathrm{Isoc}((D, Z' \cap D))$ and $N_D \in \mathrm{Hom}(\mathscr{E}_D, \mathscr{E}_D)$ is horizontal. Call N_D the *residue* of $\mathscr{E}$ along D. By definition, $\mathscr{E} \in \mathrm{Isoc}^{\mathrm{nil}}((X,Z))$ iff N_D is nilpotent for all D. This is automatic if $\mathscr{E}$ carries a Frobenius, i.e.,

$$F\text{-}\mathrm{Isoc}((X,Z)) = F\text{-}\mathrm{Isoc}^{\mathrm{nil}}((X,Z)).$$

Theorem (I, 6.4.5)

The restriction functor

$$(F\text{-})\,\mathrm{Isoc}^{\mathrm{nil}}((X,Z)) \to (F\text{-})\,\mathrm{Isoc}^{\dagger}(X \setminus Z, X)$$

is fully faithful. (This fails without requiring nilpotent residues.)

Alterations

An *alteration* $f : Y' \to Y$ is a proper, dominant, generically finite morphism. If k is perfect, we also assume f is generically étale.

Theorem (de Jong)

Let $X \subseteq Y$ be an open dense immersion of k-varieties. Then there exists an alteration $f : Y' \to Y$ such that $(Y', f^{-1}(Y \setminus X))$ is a smooth pair.

It is *not known* whether f can be taken birational over the smooth locus of Y.

de Jong's theorem stands in for resolution of singularities over k. However, knowing resolution would not improve our main theorem *except* possibly by eliminating blowups outside the regular locus.

The semistable reduction theorem

Theorem (Semistable reduction; conjectured by Shiho)

Let $X \subseteq Y$ be an open immersion of k-varieties. For $f : Y' \to Y$ an alteration, put $X' = f^{-1}(X)$ and $Z' = Y' \setminus X'$. Then for any $\mathscr{E} \in F\text{-}\mathrm{Isoc}^\dagger(X,Y)$, we can choose f so that (Y',Z') is a smooth pair and $f^*\mathscr{E}$ is the restriction (uniquely) of an element of $F\text{-}\mathrm{Isoc}^{\mathrm{nil}}((Y',Z'))$.

Semistable reduction is used by Caro and Tsuzuki to prove overholonomicity of overconvergent F-isocrystals, and by Shiho to construct generic higher direct images in relative rigid cohomology.

An analogue in characteristic 0: a higher-dimensional version of Turrittin's structure theorem for formal connections (work in progress).

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Local monodromy for isocrystals

Let (X,Z) be a smooth pair, put $U = X \setminus Z$, and let D be a component of Z. Let L be a complete discretely valued field of characteristic 0 with residue field $k(D)$, containing K with the same value group, and admitting a Frobenius lift σ extending σ_K. Note that

$$\dim_L \Omega_{L/K} = \dim(D) = \dim(X) - 1.$$

The *Robba ring* $\mathscr{R}_L$ consists of formal sums $\sum_{i \in \mathbb{Z}} c_i t^i$ with $c_i \in L$ which are convergent on some annulus $* < |t| < 1$.

For $\mathscr{E} \in \mathrm{Isoc}^\dagger(U,X)$, we obtain a *local monodromy module*, which is a finite free $\mathscr{R}_L$-module $\mathscr{E}_D$ equipped with an integrable connection

$$\nabla : \mathscr{E}_D \to \mathscr{E}_D \otimes_{\mathscr{R}_L} \Omega_{\mathscr{R}_L/K}.$$

If $\mathscr{E} \in F\text{-}\mathrm{Isoc}^\dagger(U,X)$, we also get a horizontal isomorphism $F : \sigma^* \mathscr{E}_D \cong \mathscr{E}_D$.

Unipotence and logarithmic extensions

Theorem (I, 5.2.1)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. Then the essential image of $\mathrm{Isoc}^\dagger(X,Y) \to \mathrm{Isoc}^\dagger(U,Y)$ consists of those $\mathscr{E}$ for which $\mathscr{E}_D$ is constant as a ∇-module for each codimension 1 component D of $X \setminus U$.

Theorem (I, 6.4.5)

Let (X,Z) be a smooth pair, and put $U = X \setminus Z$. Then the essential image of $\mathrm{Isoc}^{\mathrm{nil}}((X,Z)) \to \mathrm{Isoc}^\dagger(U,X)$ consists of those $\mathscr{E}$ for which $\mathscr{E}_D$ is unipotent as a ∇-module for each component D of Z.

These can be interpreted as analogues of Zariski-Nagata purity. (Does this extend to the local complete intersection case? See example of Tsuzuki.)

Some sample corollaries

Theorem (I, 5.3.1)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. For $\mathscr{E} \in \mathrm{Isoc}^\dagger(X,Y)$, any subobject of $\mathscr{E}$ in $\mathrm{Isoc}^\dagger(U,Y)$ lifts to a subobject of $\mathscr{E}$ in $\mathrm{Isoc}^\dagger(X,Y)$.

This follows because the property that $\mathscr{E}_D$ is constant passes to all subobjects. (Beware: the analogous statement for the restriction $F\text{-}\mathrm{Isoc}^\dagger(X) \to F\text{-}\mathrm{Isoc}(X)$ is false! Consider, e.g., a unit-root subcrystal.)

Theorem (I, 5.3.7)

Let $U \subseteq X \subseteq Y$ be open dense immersions with X smooth. Then

$$\mathrm{Isoc}(X) \times_{\mathrm{Isoc}^\dagger(U,X)} \mathrm{Isoc}^\dagger(U,Y) = \mathrm{Isoc}^\dagger(X,Y).$$

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The good news, and the bad news

The good news: the following is an easy consequence of "Crew's conjecture", a/k/a the *p*-adic local monodromy theorem of André-Mebkhout-K.

Theorem (Local monodromy theorem)

Let (X,Z) be a smooth pair, put $U = X \setminus Z$, and let D be a component of Z. If $\mathscr{E}_D$ carries a Frobenius structure, then there exists a finite extension $\mathscr{R}'$ of $\mathscr{R}_L$ (induced by the unramified extension of $\mathscr{R}_L^{\mathrm{bd}}$ corresponding to a finite separable extension of $k(D)((t))$) such that $\mathscr{E}_D \otimes_{\mathscr{R}_L} \mathscr{R}'$ is unipotent (as a ∇-module).

The bad news: this plus Zariski-Nagata purity is not enough to deduce semistable reduction, because an arbitrary finite extension of $k(X)$ does not correspond to a finite cover of X which is *smooth*, or even toroidal.

Krull valuations

Let X be an irreducible k-variety. A *Krull valuation* on $k(X)$ over k is a function $v : k(X) \to \Gamma \cup \{\infty\}$ for some totally ordered group Γ, such that:

- $v(x) = \infty$ iff $x = 0$;
- $v(xy) = v(x) + v(y)$;
- $v(x+y) \geq \min\{v(x), v(y)\}$.

Define

$$\begin{aligned} \Gamma_v &= v(k(X)^\times) \qquad \textit{(value group)} \\ \mathfrak{o}_v &= \{x \in k(X) : v(x) \geq 0\} \qquad \textit{(valuation ring)} \\ \mathfrak{m}_v &= \{x \in k(X) : v(x) > 0\} \qquad \textit{(maximal ideal)} \\ \kappa_v &= \mathfrak{o}_v / \mathfrak{m}_v \qquad \textit{(residue field)} \end{aligned}$$

The *center* of v on X is $\{x \in X : \mathfrak{o}_{X,x} \subseteq \mathfrak{o}_v\}$. If nonempty (e.g., if X is proper), it is closed and irreducible of dimension $\leq \operatorname{trdeg}(\kappa_v/k)$, and v is *centered on X*.

Divisorial valuations and semistable reduction

We say v is *divisorial* if v measures order of vanishing along some divisor on some variety birational to X. In particular, $\Gamma_v \cong \mathbb{Z}$.

Let $X \subseteq Y$ be an open immersion of irreducible k-varieties. For $\mathscr{E} \in \mathrm{Isoc}^\dagger(X,Y)$, we get a local monodromy module $\mathscr{E}_v$ for each divisorial valuation v on $k(X)$ centered on Y.

Theorem (approximately II, 3.4.4)

$\mathscr{E}$ admits semistable reduction if and only if there exists a finite cover $X' \to X$ with X' irreducible, such that for each divisorial valuation v on $k(X)$ centered on Y, $\mathscr{E}_v$ becomes unipotent after tensoring with the extension of $\mathscr{R}_L$ corresponding to some extension of w to $k(X')$. (Note: L is a CDVF with residue field κ_v.)

This is conveniently reformulated using Zariski-Riemann spaces.

Zariski-Riemann spaces

Let $S_{k(X)/k}$ be the set of equivalence classes of Krull valuations on $k(X)$ over k. (Here $v \sim v'$ iff $\mathfrak{o}_v = \mathfrak{o}_{v'}$.) This carries the *Zariski-Hausdorff topology*, specified by the basis of opens given by

$$\{v \in S_{k(X)/k} : v(f_1), \dots, v(f_m) \geq 0; v(g_1), \dots, v(g_n) > 0\}$$

for any $f_1, \dots, f_m, g_1, \dots, g_n \in k(X)$.

Theorem (Zariski)

The topological space $S_{k(X)/k}$ is compact.

Local semistable reduction

Let $X \subseteq Y$ be an open immersion of irreducible k-varieties. For $f : Y' \to Y$ an alteration, put $X' = f^{-1}(X)$ and $Z' = Y' \setminus X'$.

For $\mathscr{E} \in F\text{-Isoc}^\dagger(X,Y)$ and $v \in S_{k(X)/k}$ centered on Y, $\mathscr{E}$ admits *local semistable reduction at* v if there exists an alteration $f : Y' \to Y$ with Y' irreducible and an open $U \subseteq Y'$ such that $(U, U \cap Z')$ is a smooth pair, some extension of v to $k(Y')$ is centered on U, and $f^*\mathscr{E}$ lifts from $F\text{-Isoc}^\dagger(X' \cap U, U)$ to $F\text{-Isoc}^{\mathrm{nil}}((U, U \cap Z'))$.

Using Zariski's compactness theorem, we obtain the following.

Theorem (II, 3.3.4, 3.4.5; IV, 2.4.2)

Suppose that $\mathscr{E}$ admits local semistable reduction at all $v \in S_{k(X)/k}$ centered on Y. Then $\mathscr{E}$ admits semistable reduction.

Abhyankar's inequality

The *height* (*real rank*) of v is the minimum m such that Γ_v embeds into the lexicographic product $\mathbb{R}^m$.

The *rational rank* of v is $\dim_{\mathbb{Q}}(\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q})$. Note that $\mathrm{height}(v) \leq \mathrm{ratrank}(v)$.

The *corank* of v is

$$\mathrm{corank}(v) = \dim(X) - \mathrm{ratrank}(v) - \mathrm{trdeg}(\kappa_v/k).$$

Theorem (Abhyankar)

For any $v \in S_{k(X)/k}$, $\mathrm{corank}(v) \geq 0$. *Moreover, if* $\mathrm{corank}(v) = 0$, *then* $\Gamma_v \cong \mathbb{Z}^{\mathrm{ratrank}(v)}$ *and* κ_v *is finitely generated over* k.

A v with $\mathrm{corank}(v) = 0$ is called an *Abhyankar valuation*. These are dense in $S_{k(X)/k}$, since already divisorial valuations are dense.

Reductions

Theorem (II, 3.2.6)

To prove (local) semistable reduction for a given isocrystal, it suffices to do so after base change from k to k^{alg}.

Theorem (II, 4.2.4, 4.3.4)

To prove local semistable reduction over an algebraically closed field k, it suffices to do so for all valuations v with $\mathrm{height}(v) = 1$ and $\kappa_v = k$.

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Local uniformization in corank 0

Assume from now on that $k = k^{\mathrm{alg}}$. Let $X \subseteq Y$ be an open dense immersion of irreducible k-varieties. Let v be a valuation on $k(X)$ over k centered on Y with $\mathrm{corank}(v) = 0$ and $\kappa_v = k$.

Theorem (Kuhlmann, Knaf)

There is a blowup Y' of Y and local coordinates $t_1, \dots, t_n$ on Y' at the center of v, such that

$$\alpha_1 = v(t_1), \dots, \alpha_n = v(t_n)$$

are linearly independent over $\mathbb{Q}$ and generate Γ_v as a $\mathbb{Z}$-module.

For $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$, let v_β denote the $(\beta_1, \dots, \beta_n)$-Gauss valuation in terms of $t_1, \dots, t_n$. Then the completion $k(X)_v$ is isomorphic to the v_α-completion of $k[t_1^{\pm}, \dots, t_n^{\pm}]$.

Differential ramification breaks

Take $\mathscr{E} \in \mathrm{Isoc}^\dagger(X,Y)$ of rank d. Fix $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$. We may realize $\mathscr{E}$ as a ∇-module on a subspace of the $(t_1, \dots, t_n)$-affine space including (for some $\epsilon \in (0,1)$)

$$\{(t_1, \dots, t_n) : (|t_1|, \dots, |t_n|) = (\rho^{\beta_1}, \dots, \rho^{\beta_n}) \text{ for some } \rho \in (\epsilon, 1)\}.$$

Then there exist $b_1(\mathscr{E}, \beta) \geq \cdots \geq b_d(\mathscr{E}, \beta) \geq 0$ such that the intrinsic subsidiary generic radii of convergence at $(|t_1|, \dots, |t_n|) = (\rho^{\beta_1}, \dots, \rho^{\beta_n})$ are equal to $\rho^{b_1(\mathscr{E},\beta)}, \dots, \rho^{b_d(\mathscr{E},\beta)}$. These are the *differential ramification breaks* of $\mathscr{E}$ along v_β (at least if $\beta \in \mathbb{Q}^n$).

In the one-dimensional case, these are ordinary ramification breaks of the local monodromy representation (Crew, Matsuda, Tsuzuki). For more discussion, see: K, Swan conductors for p-adic differential modules, I, II.

Variation of differential Swan conductors

For $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Q}^n$, the differential ramification breaks satisfy

$$d! b_i(\mathscr{E}, \beta) \in \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_d \qquad (i = 1, \ldots, d).$$

Moreover, $b_1(\mathscr{E}, \beta) = 0$ if and only if $\mathscr{E}_{v_\beta}$ becomes unipotent after pulling back along a cover tamely ramified along $t_1 \cdots t_n = 0$ (III, 5.2.5). Define

$$B_i(\mathscr{E}, \beta) = b_1(\mathscr{E}, \beta) + \cdots + b_i(\mathscr{E}, \beta).$$

Theorem (III, 2.4.2, 4.4.7 for $i = 1$; K, Xiao in general)

The functions $d! B_i(\mathscr{E}, \beta)$ and $B_d(\mathscr{E}, \beta)$ are convex and piecewise integral affine (integral polyhedral) on $\beta \in [0, +\infty)^n$.

Two approaches to local semistable reduction

Original approach (III, 6.3.1): use an analogue of the p-adic local monodromy theorem (K, The p-adic local monodromy theorem for fake annuli) to reach a situation (after suitable alteration) where $b_1(\mathscr{E},\alpha) = 0$. Since $d!b_1(\mathscr{E},\beta) = d!B_1(\mathscr{E},\beta)$ is integral polyhedral, this forces $b_1(\mathscr{E},\beta)$ to vanish identically in a neighborhood of α.

Alternate approach (no reference yet): imitate Mebkhout's proof of the monodromy theorem, replacing Christol-Mebkhout decomposition theory with its higher-dimensional analogue (K-Xiao).

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Induction on corank

Again, assume that v is a valuation on $k(X)$ centered on v with $\kappa_v = k$, but now suppose $\mathrm{corank}(v) = m > 0$. Assume local semistable reduction for all valuations of corank $< m$.

Unfortunately, v does not admit a sufficiently convenient descriptions in local coordinates to permit an analogue of our argument in the corank 0 case. This is in part because Γ_v need not be finitely generated over $\mathbb{Z}$.

Instead, we choose a fibration $\pi : Y \to Y^0$ in curves such that $k(Y^0)$ contains a $\mathbb{Q}$-basis of $\Gamma_v \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the restriction v^0 of v to $k(Y^0)$ satisfies

$$\mathrm{corank}(v^0) = \mathrm{corank}(v) - 1.$$

A path in valuation space

Let $z \in Y$ be the center of v, and put $z_0 = \pi(z)$. Let $x \in k(X)$ restrict on $\pi^{-1}(z_0)$ to a local parameter for z.

We then identify v with a multiplicative seminorm on $k(Y^0)_{v^0}[x]$ bounded by the 1-Gauss norm, corresponding to a point of type 1 (classical) or 4 (spherical) in the Berkovich closed unit disc over $k(Y^0)_{v^0}$.

Draw the path $\mathscr{P}$ from the Gauss point to the point correponding to v. The strategy now is to reduce local semistable reduction from v to some point of $\mathscr{P} \setminus \{v\}$; any such point corresponds to a valuation with corank $m-1$, for which we are okay by the induction hypothesis.

(Aside: can we go back and formulate the whole story with Berkovich spaces instead of Zariski-Riemann spaces?)

Numerical invariants

Identify $\mathscr{P}$ with an interval $[0, s_0]$ by identifying w with $s = -\log \operatorname{radius}(w)$. (The *radius* of $w \in \mathscr{P}$ is the infimum of the radii of discs containing w.)

We define certain numerical invariants $f_1(\mathscr{E}, s) \geq \cdots \geq f_d(\mathscr{E}, s) \geq 0$ for $w \in \mathscr{P}$, akin to the differential ramification breaks (IV, 3.1.3, 5.2.1). Put $F_i(\mathscr{E}, s) = f_1(\mathscr{E}, s) + \cdots + f_i(\mathscr{E}, s)$.

Theorem (IV, 3.1.4, 5.2.3)

The $d!F_i(\mathscr{E}, s)$ are convex and piecewise affine with nonpositive integral slopes. In particular, each $f_i(\mathscr{E}, s)$ is affine in a neighborhood of v.

The hardest part is the affinity near v. Everything makes heavy use of quantitative Christol-Mebkhout theory (K, p-adic differential equations).

Warning: the $b_i(\mathscr{E}, s)$ in (IV, 4.5.1) are our $f_i(\mathscr{E}, s) + s$.

Endgame

Put $\mathscr{F} = \mathscr{E} \oplus End(\mathscr{E}) \oplus End(End(\mathscr{E}))$, where $End(\mathscr{E}) = \mathscr{E}^\vee \otimes \mathscr{E}$. By blowing up, we may reduce to the case where the $b_i(\mathscr{F}, s)$ are affine on all of $\mathscr{P}$. Then we get a "uniform Christol-Mebkhout decomposition" of $\mathscr{F}$ for all of $\mathscr{P}$, and the slope zero part becomes unipotent on a tame cover.

Using the induction hypothesis, we can pull back along an alteration to reach the case where either $b_1(\mathscr{E}, s) = 0$, or some power $\mathscr{G}$ of $\mathscr{E} \oplus End(\mathscr{E})$ locally (uniformly) admits a nonconstant rank 1 submodule whose p-th tensor power is constant. Its invariant is $b_i(\mathscr{G}, s)$ for some i, and has slopes in $\mathbb{Z}_{\leq 0}$.

One can find a Dwork isocrystal $\mathscr{L}$ such that the corresponding submodule of $\mathscr{L}^\vee \otimes \mathscr{G}$ has invariant with terminal slope ≥ 0, hence $= 0$ (IV, 5.3.1). Kill this component using the Artin-Schreier cover that trivializes $\mathscr{L}$ (plus tame). Repeat finitely many times; then $b_1(\mathscr{E}, s) = 0$ identically, and we win by the induction hypothesis.