Relative Robba rings and pushforwards in rigid cohomology

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These slides are available online at the above web site.
Also see my preprint *Finiteness of rigid cohomology with coefficients*.

Notations and conventions

k	a field of characteristic $p > 0$
K	a complete discretely valued field of characteristic 0 having residue field p
$\mathcal{O}$	the ring of integers of K
σ	an endomorphism of K lifting the p-power Frobenius on k

Context: where the Robba ring comes from

In the complex theory of differential equations with “mild” (e.g., regular) singularities, the key local object is the punctured open unit disc.

The corresponding p-adic situation arises in attempts to associate de Rham-type cohomologies to algebraic varieties in characteristic p (after lifting to a p-adic setting). Here the key local object is an open unit disc with a closed disc of *positive* radius removed. (The radius of the inner disc is tied to questions of wild ramification.) Moreover, one wants to work with “Frobenius structures” that do not preserve the radius of the inner disc.

Upshot: we typically work with germs of rigid analytic functions “at the boundary” of the open unit disc. These form the Robba ring.

The Robba ring over K

The Robba ring $\mathcal{R}_K$ consists of formal Laurent series $\sum_{n=-\infty}^{\infty} c_n t^n$ over K which satisfy the growth conditions

$$\liminf_{n\to-\infty} \frac{v_p(c_n)}{|n|} > 0, \qquad \liminf_{n\to+\infty} \frac{v_p(c_n)}{|n|} \geq 0.$$

In other words, a series belongs to $\mathcal{R}$ if and only if it converges for $t \in K^{\mathrm{alg}}$ with $\eta < |t| < 1$, for some unspecified (and not fixed) $0 < \eta < 1$. In particular, these do indeed form a ring.

Let $\mathcal{R}_K^{\mathrm{int}}$ be the subring of $\mathcal{R}_K$ consisting of series with coefficients in $\mathcal{O}$. Then $\mathcal{R}_K^{\mathrm{int}}$ is a discrete valuation ring with residue field $k((\bar{t}))$; it is henselian but not complete.

Topology of the Robba ring

The Robba ring can be viewed as the direct limit of the subrings $\mathcal{R}_\rho$ of series convergent for $\rho \leq |t| < 1$. Each $\mathcal{R}_\rho$ can in turn be viewed as the completion of $K[t, t^{-1}]$ for the Fréchet topology generated by the norms

$$\left|\sum c_n t^n\right|_\eta = \min_n\{|c_n|\eta^n\} \qquad \rho \leq \eta < 1;$$

i.e., the supremum norms on the circles $|t| = \eta$.

Theorem of Lazard: each $\mathcal{R}_\rho$, hence also $\mathcal{R}_K$, is a *Bézout ring*, that is, every finitely generated ideal is principal.

Consequence: finitely presented projective modules and finitely generated locally free modules over $\mathcal{R}_K$ are free.

Frobenius structures of differential equations

Recall that $\sigma : \mathcal{O} \to \mathcal{O}$ lifts the p-power Frobenius. Choose an extension of σ to $\mathcal{R}_K^{\text{int}}$ that does likewise; then σ extends to an endomorphism of $\mathcal{R}_K$, though not of any $\mathcal{R}_\rho$. E.g.,

$$\sum c_n t^n \mapsto \sum c_n^\sigma t^{pn}.$$

Let M be a finite (locally) free $\mathcal{R}_K$-module equipped with a connection $\nabla : M \to M \otimes \Omega^1$ (where $\Omega^1 = \Omega^{1,\text{cts}}_{\mathcal{R}_K/K}$ is free of rank 1 generated by dt). A *Frobenius structure* on M is an isomorphism

$$\sigma^* M \stackrel{\sim}{\to} M$$

of modules with connection.

Note: the definition turns out to be independent of the choice of σ.

The Robba ring and p-adic local monodromy

Let M be a finite (locally) free $\mathcal{R}_K$-module equipped with a connection $\nabla : M \to M \otimes \Omega^1$, plus a Frobenius structure. For $n \geq 0$, put

$$K_n = K^{\sigma^{-n}}$$

in case k is not perfect.

The "p-adic local monodromy theorem" (Crew's conjecture): after replacing $\mathcal{R}_K$ by

$$\mathcal{R}_{K_n} \otimes_{\mathcal{R}_{K_n}^{\mathrm{int}}} \mathcal{R}'$$

for some finite étale extension $\mathcal{R}'$ of $\mathcal{R}_{K_n}^{\mathrm{int}}$ and some n, M becomes unipotent as a module with connection. That is, M has an exhaustive ∇-stable filtration whose successive quotients are spanned by their kernels of ∇.

The Robba ring and p-adic local monodromy (cont.)

The theorem has been proven independently by André, Kedlaya, and Mebkhout.

Aside: by work of Berger, the theorem implies Fontaine's conjecture "de Rham implies potentially semistable". Moreover, the constructions in this talk are similar to those in Berger's work.

In this talk, we focus instead on consequences in p-adic cohomology.

p-adic local monodromy and cohomology

The p-adic local monodromy theorem implies the finite dimensionality of rigid (Berthelot) cohomology of *curves* with coefficients in an overconvergent F-isocrystal. (Essentially, one uses it to reduce to the case of proper curves, where finiteness in crystalline cohomology kicks in.)

To handle higher dimensional varieties, one wants to perform dévissage, i.e., fibering in curves and pushing forward to a lower dimensional variety. Thus one needs a relative version of the p-adic local monodromy theorem, formulated over a relative Robba ring.

Affinoid and dagger algebras

An *affinoid algebra* over K is a quotient of the ring of null series

$$K\langle x_1, \ldots, x_n \rangle,$$

i.e., those convergent on the closed unit polydisc.

A *dagger algebra* over K is a quotient of the ring of “overconvergent” series

$$K\langle x_1, \ldots, x_n \rangle^\dagger,$$

i.e., those convergent on some (unspecified) polydisc of radius strictly greater than 1.

A *localization* of an affinoid or dagger algebra A is a minimal algebra in the same category containing A in which some specified element of A becomes invertible.

Robba rings over affinoid algebras

Let A be a reduced affinoid algebra over K; then A carries a distinguished *spectral valuation* v_A (namely the supremum norm on $\mathrm{Maxspec}\, A$). We may then construct the Robba ring $\mathcal{R}_A$ over A by copying the definition over K: it consists of power series $\sum_{n=-\infty}^{\infty} c_n t^n$, with $c_n \in A$, such that

$$\liminf_{n \to -\infty} \frac{v_A(c_n)}{-n} > 0, \qquad \liminf_{n \to +\infty} \frac{v_A(c_n)}{n} \geq 0.$$

Equivalently, for $r > 0$ sufficiently small (depending on the series), the double sequence

$$\{c_n p^{\lfloor rn \rfloor}\}_{n \in \mathbb{Z}}$$

converges to zero in A. (This characterization can be used even if A is nonreduced.)

Relative p-adic local monodromy theorem, affinoid case

Let M be a *free* module over $\mathcal{R}_A$ equipped with a connection $\nabla : M \to M \otimes \Omega^1$ (with $\Omega^1 = \Omega^{1,\text{cts}}_{\mathcal{R}_A/A}$ generated by dt) and a Frobenius structure. For any localization B of A and $n \geq 0$, put $B_n = B^{\sigma^{-n}}$.

Then for some localization B of A, we can tensor up to $\mathcal{R}_{B_n} \otimes_{\mathcal{R}^{\text{int}}_{B_n}} \mathcal{R}'$ for some finite étale extension $\mathcal{R}'$ of $\mathcal{R}^{\text{int}}_{B_n}$ and some $n \geq 0$, such that M becomes unipotent as a module with connection.

The trouble: this only allows us to construct higher direct images of families of curves as *convergent* F-isocrystals. To continue the dévissage, we need to do likewise in the *overconvergent* category.

Robba rings over dagger algebras, attempt 1

What is the “correct” definition of the Robba ring over a dagger algebra A?

Guess 1: the subring of $\mathcal{R}_{\widehat{A}}$ of series with coefficients in A.

Problem: that’s not a ring!

Interlude: fringe subalgebras

For any $\rho > 1$ in the norm group of K^{alg}, let

$$K\langle x_1, \ldots, x_n \rangle_\rho$$

denote the ring of formal power series in $x_1, \ldots, x_n$ convergent for $|x_i| \leq \rho$. This is an affinoid algebra, and $K\langle x_1, \ldots, x_n \rangle^\dagger$ is the union of these algebars over all $\rho > 1$.

Now suppose A is a dagger algebra. For any presentation

$$K\langle x_1, \ldots, x_n \rangle^\dagger \twoheadrightarrow A$$

and any $\rho > 1$ in the norm group of K^{alg}, the image of $K\langle x_1, \ldots, x_n \rangle_\rho$ is an affinoid algebra contained in A. We call such a subalgebra a *fringe (sub)algebra*.

Interlude: fringe subalgebras (cont.)

Upshot: a dagger algebra A can be viewed as a direct limit of affinoid subalgebras, namely its fringe subalgebras. This yields a natural topology on A, its *dagger topology*, which is finer than the “affinoid topology” induced by the spectral seminorm.

Robba rings over dagger algebras, attempt 2

What is the “correct” definition of the Robba ring over a dagger algebra A?

Guess 2: take the direct limit of $\mathcal{R}_B$ over all fringe subalgebras B of A.

Problem: this is too small to be useful. For instance, if $A = K\langle x\rangle^\dagger$, then $1 - xt$ does not have an inverse in this ring.

Robba rings over dagger algebras, attempt 3

What is the "correct" definition of the Robba ring over a dagger algebra A?

Guess 3 (the right one): it consists of series $\sum_{n \in \mathbb{Z}} c_n t^n$, with $c_n \in A$ for all n, such that for $r > 0$ sufficiently small, the double sequence

$$\{c_n p^{\lfloor rn \rfloor}\}_{n \in \mathbb{Z}}$$

converges *in some fringe subalgebra of* A (depending on r).

Robba rings over dagger algebras, attempt 3 (cont.)

Example: if $A = K\langle x \rangle^\dagger$, a series

$$\sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} x^m t^n$$

belongs to $\mathcal{R}_A$ if and only if for some $\delta > 1$ and some $\epsilon > 0$, the series converges for

$$|x| \leq \delta, \qquad \delta^{-1} \leq |t| < 1, \qquad |x^\epsilon t| < 1.$$

Relative p-adic local monodromy theorem, dagger case

Let M be a *free* module over $\mathcal{R}_A$ equipped with a connection $\nabla : M \to M \otimes \Omega^1$ (with $\Omega^1 = \Omega^{1,\mathrm{cts}}_{\mathcal{R}_A/A}$ generated by dt) and a Frobenius structure. For any localization B of A and $n \geq 0$, put $B_n = B^{\sigma^{-n}}$.

Then for some localization B of A, we can tensor up to $\mathcal{R}_{B_n} \otimes_{\mathcal{R}^{\mathrm{int}}_{B_n}} \mathcal{R}'$ for some finite étale extension $\mathcal{R}'$ of $\mathcal{R}^{\mathrm{int}}_{B_n}$ and some $n \geq 0$, such that M becomes unipotent as a module with connection.

Point: with the correct definition of $\mathcal{R}_A$, the theorem admits the same formulation as in the simpler affinoid case!

Comments on the proof

The key step is to show that if $M \otimes \mathcal{R}_L$ is unipotent, for L the completed fraction field of A, then so is $M \otimes \mathcal{R}_B$ for some localization B of A. We do this by constructing an iteration that converges to a nonzero horizontal element of M.

For instance, suppose $M \otimes \mathcal{R}_L$ admits a basis of horizontal sections. Let $D : M \to M$ be the operator defined by

$$\nabla \mathbf{v} = D\mathbf{v} \otimes \frac{dt}{t}.$$

Put

$$f_n(\mathbf{v}) = \frac{(D^2 - 1^2)(D^2 - 2^2) \cdots (D^2 - n^2)}{(n!)^2} \mathbf{v};$$

then $\{f_n(\mathbf{v})\}$ converges to a horizontal element of M.

Comments on the proof (cont.)

Where does the localization come in? It is needed to ensure that the extension of the Robba ring is again a Robba ring. For instance, if $A = K\langle x \rangle^\dagger$, then the extension

$$\mathcal{R}_A[z]/(z^p - z - \frac{x}{t})$$

cannot be presented as a Robba ring.

On the other hand, if B is a localization of A in which x becomes invertible, then

$$\mathcal{R}_B[z]/(z^p - z - \frac{x}{t})$$

is isomorphic to $\mathcal{R}_B$ with series parameter z^{-1}.

Application: rigid cohomology

Let A be a dagger algebra of the form

$$A^{\mathrm{int}} \otimes_{\mathcal{O}} K = (\mathcal{O}\langle x_1, \dots, x_n \rangle^{\dagger}/I) \otimes_{\mathcal{O}} K,$$

where A^{int} is flat over $\mathcal{O}$ and $A^{\mathrm{int}} \otimes_{\mathcal{O}} k$ is smooth over k. We say that A is *of MW-type* (for Monsky-Washnitzer), corresponding to the smooth affine k-variety $X = \mathrm{Spec}(A^{\mathrm{int}} \otimes_{\mathcal{O}} k)$.

Note: A is determined by X up to noncanonical isomorphism, but the (continuous) de Rham complex of A is determined up to homotopy isomorphism.

Let M be a finite locally free A-module equipped with a connection $\nabla : M \to \Omega^1$ plus a Frobenius structure. That is, M is an *overconvergent F-isocrystal* over A.

Application: rigid cohomology (cont.)

Theorem: The cohomology of the de Rham complex $M \otimes \Omega^\cdot$ is finite dimensional. It constitutes the rigid cohomology of $\mathrm{Spec}(A^{\mathrm{int}} \otimes_{\mathcal{O}} k)$ with coefficients in M.

The case $M = A$ is due to Berthelot. For the general case, we perform dévissage, by constructing "generic" higher direct images for families of curves; see next slide for the key case.

Note "generic" restriction: we are working with an analogue of lisse sheaves, not constructible sheaves, so we cannot expect to have Grothendieck's six operations. For that one must pass to a suitable $\mathcal{D}$-module category, which is not yet fully understood.

Application: some pushforwards in rigid cohomology

Put $B = A\langle y\rangle^\dagger$; that is, if $A = K\langle x_1, \dots, x_n\rangle^\dagger/I$, put

$$B = K\langle x_1, \dots, x_n, y\rangle^\dagger/(IK\langle x_1, \dots, x_n, y\rangle^\dagger).$$

Let M be an overconvergent F-isocrystal over B. Let ∇_y be the component of ∇ in the direction of dy.

Theorem: after replacing A by a localization, the kernel and cokernel of ∇_y become finitely generated locally free A-modules.

Idea: reduce to the case where A is a field (studied by Crew) by embedding B into $\mathcal{R}_A$ (with $t = y^{-1}$), and analyze local cohomology using the relative p-adic local monodromy theorem.

Other applications

1. Establish an analogue of Deligne's "Weil II" (and recover the Weil conjectures) by purely p-adic methods (see web site).

2. Exhibit "semistable reduction" for overconvergent F-isocrystals: show that every overconvergent F-isocrystal on a k-variety, after pullback along a suitable alteration (as per de Jong), extends to a log-F-isocrystal on a good compactification (in progress).

3. Construct higher direct images for more general families, towards a conjecture of Berthelot: for $X \to Y$ smooth proper, any overconvergent F-isocrystal on X admits higher direct images on Y (not yet in progress).