The differential Swan conductor

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The Robba ring

Let K be a complete discretely valued field of mixed characteristic $(0,p)$, with ring of integers $\mathfrak{o}_K$, maximal ideal $\mathfrak{m}_K$, and residue field k. Let $A_K(\rho,1)$ be the annulus $\rho < |t| < 1$. Define the *Robba ring*

$$\mathscr{R} = \bigcup_{\rho \in (0,1)} \Gamma(A_K(\rho,1), \mathscr{O}).$$

Write elements of $\mathscr{R}$ as formal Laurent series $\sum c_i t^i$. The subring

$$\mathscr{R}^{\mathrm{int}} = \{\sum c_i t^i \in \mathscr{R} : c_i \in \mathfrak{o}_K \quad (i \in \mathbb{Z})\}$$

is a henselian (noncomplete) discrete valuation ring with residue field $k((t))$.

The p-adic local monodromy theorem

Let q be a power of p. Pick a map $\sigma : \mathscr{R}^{\text{int}} \to \mathscr{R}^{\text{int}}$ of the form

$$\sum c_i t^i \mapsto \sum \sigma_K(c_i) u^i,$$

where σ_K lifts the absolute q-power Frobenius, and $u \equiv t^q \pmod{\mathfrak{m}_K}$. Let M be an (F, ∇)-*module*, i.e., a finite free $\mathscr{R}$-module equipped with:

- a connection $\nabla : M \to M \otimes \Omega^1_{\mathscr{R}/K} = M \otimes dt$;
- an isomorphism $F : \sigma^* M \to M$ of modules with connection.

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Theorem (André, K, Mebkhout)

There exists a finite étale extension $\mathscr{S}$ of $\mathscr{R}^{\mathrm{int}}$ such that as a module with connection, $M \otimes_{\mathscr{R}^{\mathrm{int}}} \mathscr{S}$ is a successive extension of trivial modules (i.e., M is quasi-unipotent).

Monodromy representations

Write $G_F = \mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Corollary

The category of $\mathscr{R}$-modules with quasi-unipotent connection is equivalent to the category of semilinear representations of

$$G = G_{k((t))} \times K$$

on finite dimensional K^{unr}-vector spaces, which are potentially trivial on the first factor and unipotent on the second factor.

In particular, the restriction to the inertia subgroup of $G_{k((t))}$ descends to a linear representation with finite image, so has a well-defined Swan conductor.

p-adic irregularity

For M an (F,∇)-module, one can define a p-adic analogue of the *irregularity* of a meromorphic connection on a complex analytic space.

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Theorem (André, Crew, Matsuda, Tsuzuki)

The p-adic irregularity of M agrees with the Swan conductor of the monodromy representation.

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- *Relate this to the Abbes-Saito logarithmic conductor.*
- Study the variation of p-adic irregularity for a fixed overconvergent F-isocrystal *or a lisse ℓ-adic sheaf* on a fixed surface, but varying the choice of a boundary divisor.
- Apply this to the problem of semistable reduction for overconvergent F-isocrystals on surfaces *and higher-dimensional varieties*.

Contents

Setup

Let L/K be an extension of complete discretely valued fields of mixed characteristics $(0,p)$, such that L and K have the same value group, and the module of continuous differentials $\Omega^1_{L/K}$ admits a finite basis $dx_1,\dots,dx_n$. (E.g., take L to be the completion of $K(x_1,\dots,x_n)$ for the $(1,\dots,1)$-Gauss norm.)

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On the annulus $A_L(\varepsilon,1)$, we can construct a sheaf $\Omega^1_{A_L(\varepsilon,1)/K}$ of continuous differentials *relative to* K.

Let $\mathscr{E}$ be a coherent locally free $\mathscr{O}$-module on $A_L(\varepsilon,1)$, equipped with an integrable K-linear (but not L-linear) connection

$$\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{A_L(\varepsilon,1)/K}.$$

(The integrability is a nonempty condition as soon as $n > 0$.)

Generic radii of convergence (after Christol-Dwork)

For $\rho \in (\varepsilon, 1)$, let F_ρ be the completion of $L(t)$ for the ρ-Gauss norm. View F_ρ as a differential field of order $n+1$, for the derivations

$$\partial_1, \dots, \partial_{n+1} = \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}.$$

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Put $\mathscr{E}_\rho = \mathscr{E} \otimes F_\rho$, viewed as a differential module over F_ρ. Equip $\mathscr{E}_\rho$ with any norm compatible with the norm on F_ρ; these are all equivalent. Thus, although the *operator norm*

$$|\partial_i|_{\mathscr{E}_\rho} = \sup_{\mathbf{v} \in \mathscr{E}_\rho, |\mathbf{v}|=1} |\partial_i(\mathbf{v})|$$

depends on the choice of the norm, the *spectral norm* is well-defined:

$$|\partial_i|_{\mathscr{E}_\rho, \mathrm{sp}} = \lim_{n \to \infty} |\partial_i^n|_{\mathscr{E}_\rho}^{1/n}.$$

Generic radii of convergence (continued)

Define the *scale* of ∂_i on $\mathscr{E}_\rho$ as

$$s_i = \frac{|\partial_i|_{\mathscr{E}_\rho,\mathrm{sp}}}{|\partial_i|_{F_\rho,\mathrm{sp}}};$$

the denominator is $|p|^{1/(p-1)}$ for $i = 1, \dots, n$ and $\rho^{-1}|p|^{1/(p-1)}$ for $i = n+1$.

Generic radii of convergence (continued)

Define the *scale* of ∂_i on $\mathscr{E}_\rho$ as

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the denominator is $|p|^{1/(p-1)}$ for $i = 1, \dots, n$ and $\rho^{-1}|p|^{1/(p-1)}$ for $i = n+1$. Define the *(toric) generic radius of convergence* of $\mathscr{E}_\rho$ as

$$T(\mathscr{E}, \rho) = \min\{s_1^{-1}, \dots, s_{n+1}^{-1}\}.$$

In the case $n = 0$, $\rho \cdot T(\mathscr{E}, \rho)$ is the radius of the largest open disc around a "generic" point of $A_L(\rho, 1)$ at distance ρ from the origin, on which $\mathscr{E}$ admits a basis of horizontal sections.

Note: a beautiful coordinate-free definition has been given by Baldassarri.

Connections solvable at 1

Theorem (after Christol-Dwork, Christol-Mebkhout)

The function $f : (0, -\log \varepsilon) \to \mathbb{R}$ *given by* $f(r) = \log T(\mathscr{E}, e^{-r})$ *is concave and piecewise linear, with slopes in* $(\operatorname{rank} \mathscr{E})!^{-1}\mathbb{Z}$*.*

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We say $\mathscr{E}$ is *solvable at* 1 if

$$\lim_{\rho \to 1^-} T(\mathscr{E}, \rho) = 1.$$

This is automatic if $\mathscr{E}$ admits a Frobenius structure.

For $\mathscr{E}$ solvable at 1, there exists some $b \in \mathbb{R}_{\geq 0}$ such that $T(\mathscr{E}, \rho) = \rho^b$ for $\rho \in (\varepsilon, 1)$ sufficiently close to 1. We call b the *differential highest ramification break* of $\mathscr{E}$.

Example: Dwork isocrystals

Assume $\pi \in K$ with $\pi^{p-1} = -p$. Let L be the completion of $K(x)$ for the 1-Gauss norm. Put $\mathscr{E} = \mathscr{O}\mathbf{v}$ with

$$\nabla(\mathbf{v}) = \pi \mathbf{v} \otimes d(x^a t^{-b})$$

with $a \in \mathbb{Z}$, $b \in \mathbb{Z}_{>0}$, and a, b not both divisible by p. Then the differential highest break of $\mathscr{E}$ is equal to b.

This construction is analogous to Artin-Schreier sheaves in the étale setting. In fact, this is better than just an analogy!

The Hasse-Arf polygon

Assume that $\mathscr{E}$ is solvable at 1.

Theorem (after Christol-Mebkhout)

Over $A_L(\delta,1)$ for some δ, there exists a direct sum decomposition $\mathscr{E} = \oplus_b \mathscr{E}_b$ such that for any $\rho \in (\delta,1)$, every constituent of $\mathscr{E}_{b,\rho}$ (as a differential module over F_ρ) has maximum scale ρ^{-b}.

(In case $n = 0$, *every* nonzero local horizontal section of $\mathscr{E}_b$ around a generic point at distance ρ from the origin has radius of convergence ρ^{b+1}.)

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(In case $n = 0$, *every* nonzero local horizontal section of $\mathscr{E}_b$ around a generic point at distance ρ from the origin has radius of convergence ρ^{b+1}.)

The *Hasse-Arf polygon* of $\mathscr{E}$ is defined to have slope b with multiplicity $\operatorname{rank}(\mathscr{E}_b)$. The *differential Swan conductor* of $\mathscr{E}$ is

$$\operatorname{Swan}(\mathscr{E}) = \sum_b b \operatorname{rank}(\mathscr{E}_b).$$

The Hasse-Arf property

Assume that $\mathscr{E}$ is solvable at 1.

Theorem

Swan$(\mathscr{E})$ *is a nonnegative integer.*

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Theorem

$\mathrm{Swan}(\mathscr{E})$ *is a nonnegative integer.*

The proof uses Newton polygons for twisted polynomials over differential fields. It does not provide a cohomological interpretation; that is, I do not know how to exhibit $\mathrm{Swan}(\mathscr{E})$ as the dimension of a naturally arising vector space (except in the case $n = 0$).

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Setup

Let E be a complete discretely valued field of characteristic $p > 0$, with residue field k. Assume for simplicity that $[k : k^p] = p^n < \infty$, although the results can be extended to the general case.

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By a *representation* of G_E, we will mean a continuous homomorphism $\rho : G_E \to \mathrm{GL}(V)$, for V a finite dimensional vector space over some finite extension F of $\mathbb{Q}_p$.

We will be interested in representations of G_E with *finite local monodromy*, i.e., such that the restriction of ρ to the inertia group I_E has finite image.

From representations to differential modules

Apply Cohen's structure theorem to write $E \cong k((t))$, so as to identify E with the residue field of $\mathscr{R}^{\mathrm{int}}$.

An (F,∇)-module over $\mathscr{R}$ is *unit-root* (*étale*) if it arises by extension of scalars from an (F,∇)-module over $\mathscr{R}^{\mathrm{int}}$.

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Theorem (after Tsuzuki)

There is an equivalence of categories between representations of G_E with finite local monodromy, and (F,∇)-modules over $\mathscr{R}$ with unit-root Frobenius structure.

We can use this to define differential Swan conductors for representations of G_E with finite local monodromy. (The roles of L, K before are now played by K and a subfield K_0 whose residue field is the maximal perfect subfield of k.)

For instance, Dwork isocrystals arise from Artin-Schreier characters.

Transfer from ℓ-adic to p-adic

Let W be a finite dimensional vector space over some finite extension F of $\mathbb{Q}_\ell$, for $\ell \neq p$ a prime, and let $\rho : G_E \to \mathrm{GL}(W)$ be a continuous homomorphism. Then ρ is necessarily quasi-unipotent, so the semisimplification $(\rho|_I)^{\mathrm{ss}}$ factors through the quotient H by a cofinite open subgroup.

The representation $(\rho|_I)^{\mathrm{ss}}$ of the finite group H can be defined over the field $\mathbb{Q}^{\mathrm{ab}}$, which can be embedded into a finite extension of $\mathbb{Q}_p$. We can thus construct a p-adic representation of H, and hence obtain a differential Swan conductor.

Similarly, we can define differential Swan conductors for representations with finite local monodromy into vector spaces over any field of characteristic 0.

Comparison with Abbes-Saito?

Abbes-Saito defined a *logarithmic filtration* on G_E, which agrees with the upper numbering filtration if $n = 0$. This gives another notion of conductor for representations with finite local monodromy.

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Problem (after Matsuda)

Does this agree with the differential Swan conductor? (Progress by Matsuda, L. Xiao.)

An affirmative answer would imply that the Abbes-Saito conductor is an integer, which is otherwise not known in general.

The Abbes-Saito construction also works if E has mixed characteristic. I don't know whether it has a differential interpretation in that case, or whether one can prove integrality of conductors.

Contents

Setup

Let X be a smooth variety over k. An *overconvergent F-isocrystal* on X is the p-adic analogue of a lisse ℓ-adic sheaf on X.

Embed X into a proper variety $\overline{X}$. Locally, lift $\overline{X}$ to a formal scheme P over $\mathfrak{o}_K$ smooth near X. An overconvergent F-isocrystal consists of a module with integrable connection on a strict neighborhood of the tube $]X[$, plus an action of a lift of Frobenius.

Conductors along divisors

Let $\mathscr{E}$ be an overconvergent F-isocrystal on a smooth irreducible variety X. Given any irreducible divisor $Z \subset \overline{X}$, we can "restrict $\mathscr{E}$ to the generic point of Z", obtaining an (F, ∇)-module over $\mathscr{R}_L$ with L a complete discretely valued field with residue field $k(Z)$.

This leads to a differential Swan conductor $\mathrm{Swan}(\mathscr{E}, Z)$ along Z. It is more canonical to view the construction as taking as input an overconvergent F-isocrystal plus a divisorial valuation on $k(X)$. (It will then extend continuously to certain non-divisorial valuations.)

Normalization

In its natural normalization, $\mathrm{Swan}(\mathscr{E}, Z)$ is always a nonnegative integer. To state continuity results, it is better to normalize in terms of a fixed function $t \in k(X)$.

If Z corresponds to the surjective divisorial valuation $v : k(X) \to \mathbb{Z}$, put

$$\mathrm{Swan}_t(\mathscr{E}, Z) = \frac{\mathrm{Swan}(\mathscr{E}, Z)}{|v(t)|}.$$

Convexity

Let X be the complement in a smooth variety $\overline{X}$ of a strict normal crossings divisor $D = D_1 \cup \cdots \cup D_m$. Let $t_1, \dots, t_m$ be local parameters for $D_1, \dots, D_m$ at some point $x \in D$ where they meet. For $r = (r_1, \dots, r_{m-1}) \in \mathbb{Q}_{\geq 0}^{m-1}$, define the valuation

$$v_r : t_1 \sim t_m^{r_1}, \dots, t_{m-1} \sim t_m^{r_{m-1}}.$$

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Theorem

Let $\mathscr{E}$ be an overconvergent F-isocrystal on X. Then $\mathrm{Swan}_{t_m}(\mathscr{E}, r)$ extends continuously to $\mathbb{R}_{\geq 0}^{m-1}$, and is convex and piecewise of the form

$$\mathrm{Swan}_{t_m}(\mathscr{E}, r) = c_1 r_1 + \cdots + c_{m-1} r_{m-1} + d \qquad (c_1, \ldots, c_{m-1}, d \in \mathbb{Z}).$$

Subharmonicity and monotonicity

Assume $k = k^{\mathrm{alg}}$. Let $\overline{X}$ be a smooth proper irreducible surface, and let Z be a curve in $\overline{X}$. Let $\mathscr{E}$ be an overconvergent F-isocrystal on some open dense subscheme of $\overline{X}$. For $z \in Z$, let t, x be local coordinates at z with $Z = V(t)$. Let $\mathrm{Swan}'(\mathscr{E}, z)$ be the derivative as $r \to 0^+$ of the conductor of $\mathscr{E}$ along $v_{r,z} : x \sim t^r$, normalized with respect to t.

Subharmonicity and monotonicity

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Theorem

There exists $\ell \in \{0, \ldots, \mathrm{rank}(\mathscr{E})\}$ *such that:*

(a) *we have* $\mathrm{Swan}'(\mathscr{E}, z) + \ell = 0$ *for all but finitely many* z*;*

(b) *we have* $\mathrm{Swan}'(\mathscr{E}, z) + \ell \leq 0$ *if* $\mathscr{E}$ *is defined on the complement of* Z *in a neighborhood of* z*;*

(c)
$$\sum_{z \in Z} (\mathrm{Swan}'(\mathscr{E}, z) + \ell) \geq (2 - 2g(Z))\ell - Z^2\,\mathrm{Swan}(\mathscr{E}, Z).$$

Interpretation on Berkovich space

Keep notation from the previous slide. The set of valuations centered on z form (almost) a Berkovich analytic space. The above results can be interpreted as potential-theoretic properties of the conductor function (à la Thuillier) *except* that they are only defined for the metric topology, not the Berkovich topology.

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The main problem is that one can have $\ell \neq 0$. E.g., for a Dwork isocrystal

$$\nabla(\mathbf{v}) = \pi \mathbf{v} \otimes d(x^a t^{-b})$$

in the (x,t)-plane, along $t = 0$ we have $\ell = 0, 1$ according as b is not, is divisible by p.

These properties are needed for semistable reduction for overconvergent F-isocrystals on surfaces (specifically, for local semistable reduction at infinitely singular valuations with non-finitely generated value group).

Transfer from ℓ-adic to p-adic?

Again, we can define a conductor for a lisse ℓ-adic sheaf on a variety, measured along a boundary divisor. It is much less clear how to transfer p-adic results, but we expect it to be possible.

This would imply good variational properties of the Abbes-Saito logarithmic conductor in equal characteristic.

Complex analogues?

There are some questions about irregularity of meromorphic connections on complex analytic varieties of dimension greater than 1, that seem analogous to what we consider here. Can these ideas be transferred to the (probably simpler) setting over $\mathbb{C}$?

The end

Thank you. (Arigatō gozaimashita.)