Convergence of solutions of p-adic differential equations and higher-dimensional ramification theory

Kiran S. Kedlaya (MIT)

kedlaya@mit.edu

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Overview: Hodge theory in char. p

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- In this correspondence, wild ramification corresponds to failure of convergence of certain local horizontal sections, and to nonvanishing of irregularity of connections.
- Goal here: extend this correspondence to p-adic representations of, e.g., $\pi_1^{\mathrm{et}}(\operatorname{Spec} k[[x,y]][1/x,1/y])$.

More context

This arises in the context of a conjecture of Shiho, on constructing logarithmic extensions of overconvergent F-isocrystals on k-varieties; one wants to measure variation of "p-adic local monodromy" on a surface as one varies the choice of a boundary divisor.

It seems to be closely analogous to conjectures of Sabbah (currently being considered by Y. André) on variation of irregularity of an algebraic connection on a surface, again as one varies the choice of a boundary divisor.

Notation

- $k :=$ an algebraically closed field of char. $p > 0$
- $K := \mathrm{Frac}\, W$ for $W = W(k)$
- $q :=$ a fixed power of p
- $\sigma :=$ the q-power Frobenius on k, W, K
- $G_{k((t))} := \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t)))$
- $A(r, 1) :=$ the rigid analytic annulus $r < |t| < 1$ over K
- $\mathcal{R} := \cup_{0<r<1} \Gamma(A(r, 1), \mathcal{O})$ (the *Robba ring*)

p-adic differential modules

Extend σ to $\mathcal{R}$, e.g., by setting $\sigma(t) = t^q$. A *Frobenius action* on a module with integrable connection (∇-module) $\mathcal{E}$ over $\mathcal{R}$ is an isomorphism $F : \sigma^*\mathcal{E} \cong \mathcal{E}$.

Theorem (André, K, Mebkhout). *There is an equivalence of categories*

$$\left\{\begin{matrix}\text{discrete-unipotent reps} \\ G_{k((t))} \times K \to \mathrm{GL}_*(K)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\nabla\text{-modules on } \mathcal{R} \text{ admit-} \\ \text{ting a Frobenius action}\end{matrix}\right\}.$$

Here discrete reps of $G_{k((t))}$ correspond to *étale* ∇-modules, a/k/a those admitting unit-root Frobenius actions (Tsuzuki).

Generic radii of convergence

Let $\mathcal{E}$ be a ∇-module over $\mathcal{R}$. For $r \in (0,1)$ where $\mathcal{E}$ is defined, consider any t_r with $|t_r| = r$ in any complete extension K' of K. Compute the supremum of those $\lambda \leq r$ for which $\mathcal{E}$ has a basis of horizontal sections on the disc

$$\{t \in A(r,1) : 0 < |t - t_r| < \lambda\};$$

let $R(\mathcal{E}, r)$ be the infimum over all choices of K', t_r.

$R(\mathcal{E}, r) =$ *generic radius of convergence* of $\mathcal{E}$ at r (Christol-Dwork).

An example

Suppose $\pi^{p-1} = -p$. Pick a positive integer m coprime to p. Define $\mathcal{E}$ of rank 1:

$$\nabla \mathbf{v} = m\pi t^{-m-1}\mathbf{v} \otimes dt.$$

Then a horizontal section around t_r is

$$\exp \pi(t^{-m} - t_r^{-m})\mathbf{v},$$

which converges for $|t^{-m} - t_r^{-m}| < 1 \Leftrightarrow |t - t_r| < r^{1+m}$, so

$$R(\mathcal{E}, r) = r^{1+m}.$$

Ramification and convergence

Let $\mathcal{E}$ be a ∇-module over $\mathcal{R}$. We say $\mathcal{E}$ has *highest break* β if $R(\mathcal{E}, r) = r^{1+\beta}$ for $r \in (0,1)$ close enough to 1.

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Theorem (André, Christol-Mebkhout, Crew, K, Matsuda, Tsuzuki)**.** *If $\mathcal{E}$ is a ∇-module over $\mathcal{R}$ admitting a Frobenius structure, then it has highest break equal to the highest ramification break of the corresponding $G_{k((t))} \times K$-representation (ignoring the K).*

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For instance, the example on the previous slide corresponds to a nontrivial character of

$$k((t))[z]/(z^p - z - t^{-m})$$

which indeed has highest break m when m is coprime to p.

Swan conductors and irregularity

Swan conductors on the representation side correspond to certain sums of highest breaks on the differential side; Christol-Mebkhout showed these can be interpreted as irregularities of p-adic connections in the sense of Robba. (These can be computed from algebraic irregularities by adding certain p-adic correction terms.)

This yields a Grothendieck-Ogg-Shafarevich formula in rigid cohomology, by showing that the p-adic corrections sum to a discrepancy between Euler-Poincaré characteristics in algebraic and p-adic settings. This uses both complex and rigid GAGA!

A higher-dimensional situation

Consider a ∇-module $\mathcal{E}$ on the rigid space

$$\{\rho_x < |x| < 1, \rho_y < |y| < 1\} \subset \mathbb{A}^2_K.$$

Assume hereafter that $\mathcal{E}$ admits a Frobenius action $\sigma^*\mathcal{E} \cong \mathcal{E}$, where σ is extended to an action on $\mathbb{A}^n_K$ via $x \mapsto x^q, y \mapsto y^q$.

The subcategory of such $\mathcal{E}$ admitting unit-root Frobenius actions is equivalent to discrete K-linear representations of $\pi_1(\operatorname{Spec} k[[x,y]][1/x, 1/y])$. It is unclear what such statement could be made about the whole category.

Generic radii of convergence, again

For $R = (r_x, r_y) \in (\rho_x, 1) \times (\rho_y, 1)$, consider any x_R, y_R of norms r_x, r_y in any complete extension K' of K. Compute the supremum of those $\lambda \leq 1$ for which $\mathcal{E}$ has a basis of horizontal sections on

$$|x - x_R| < \lambda r_x, |y - y_R| < \lambda r_y,$$

then let $T(\mathcal{E}, R)$ be the infimum over all choices of K', x_R, y_R. This function is convex in R, hence continuous.

Here T stands for “toric normalization”: this construction commutes in a suitable sense (for R close to 1) with blowups at $(0, 0)$, e.g., $x \mapsto xy$.

An example

Let $\mathcal{E}$ be the rank 1 ∇-module:

$$\nabla \mathbf{v} = \pi \mathbf{v} \otimes d(x^{-e} y^{-f})$$

where at least one of e, f is not divisible by p. Then

$$T(\mathcal{E}, R) = \min\{1, r_x^e r_y^f\}.$$

This corresponds to a character of the extension defined by $z^p - z = x^{-e} y^{-f}$.

Rationality

For $\mathcal{E}, r_x, r_y$ fixed and $c \to 0^+$, we have

$$\log T(\mathcal{E}, (r_x^c, r_y^c)) = \frac{c}{m}(a \log(r_x) + b \log(r_y))$$

for some $a, b \in \mathbb{Z}$ depending on $\mathcal{E}, r_x, r_y$ only, and some $m \in \mathbb{Z}_{>0}$ depending on $\mathrm{rank}(\mathcal{E})$ only.

Moreover, a, b are piecewise constant as a function of $l = \log(r_y)/\log(r_x)$.

Highest breaks

We may think of

$$\lim_{c \to 0^+} \frac{\log T(\mathcal{E}, (r_x^c, r_x^{lc}))}{c \log(r_x)}$$

as a “highest break” of $\mathcal{E}$ along “the divisor $y \sim x^l$”. This makes sense for $l \in \mathbb{Q}$ (write $l = r/s$ and consider the exceptional divisor where $x^r \sim y^s$), but also for $l \notin \mathbb{Q}$ using an analogue of the AKM theorem for “fake annuli”.

Swan conductors

By adding highest breaks, we get a “Swan conductor” for $\mathcal{E}$ along $y \sim x^l$. Warning: continuity of this function is not yet obvious; it will follow from the harmonicity result to follow.

For $l \in \mathbb{Q}$, the Swan conductor has denominator bounded by the denominator of l times some constant.

Question (Hasse-Arf problem)**.** *Is that constant* 1*?*

Maybe one can answer this by giving a cohomological interpretation (via a higher-dimensional version of Robba irregularity)?

Reconciliation (after Matsuda)

Say we start with a representation of

$$\pi_1^{\mathrm{et}}(\operatorname{Spec} k[[x,y]][1/x,1/y])$$

and convert into a ∇-module $\mathcal{E}$. For $l = r/s \in \mathbb{Q}$, compute the “highest break” of the corresponding rep of G_F for $F = k(x^r/y^s)((t))$, where $t = x^u y^v$ with $ru + sv = 1$. (Reminder: F is a local field with *imperfect* residue field, so usual ramification theory does not apply.)

Question. *Is this consistent with Abbes–T. Saito's definition of highest breaks? (Yes for Artin-Schreier.) Or Zhukov's definition over* $k((x^r/y^s))((t))$*?*

Harmonicity

One can define Swan conductors on (the interior of) the Berkovich affine line over $k((x))$; working along $y \sim x^l$ corresponds to looking at the generic point of the disc $|y| = |x|^l$. This space is an “infinitely branched tree” and one can define *harmonic functions* on it; see the Rennes PhD thesis of A. Thuillier.

Rough explanation: at any point, the function is linear with the same slope along all but finitely many of the branches, and the slopes along the other branches average to this common value.

Harmonicity (contd.)

Proposition. *The highest break and Swan conductor are harmonic functions on the Berkovich line; in particular, they are continuous.*

Idea of proof: use Frobenius antecedent theorem (Christol-Dwork) to read the highest break off of a certain Newton polygon. (Compare: the proof of existence of the highest break in the original situation of Christol-Mebkhout.)

Exercise: do this directly for ∇-modules corresponding to Artin-Schreier characters.

Deligne-Laumon semicontinuity

Proposition. *Say $\mathcal{E}$ is actually defined on*

$$\{(x,y) \in \mathbb{A}^2_K : |x| \leq 1, \rho < |y| \leq 1\}$$

(e.g., $\mathcal{E}$ arises from an overconvergent F-isocrystal on $\mathbb{A}^1_k \times \mathbb{G}_{m,k}$). Then the highest break along $y \sim x^l$ is a decreasing *function of l.*

This follows from a version of semicontinuity in rigid cohomology, with a similar proof (using a vanishing cycles construction, as in the proof of rigid “Weil II”).

Fini