Irregularity of flat meromorphic connections

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References: Good formal structures for flat meromorphic connections, I (*Duke Math. J.*, 2010), II (*JAMS*, 2011).

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Flat meromorphic connections

Throughout this talk, let K be a field of characteristic 0. You may safely assume K is algebraically closed or even $K = \mathbb{C}$.

Let X be a smooth irreducible variety over K. Let Z be a closed proper subset and put $U = X - Z$. A *flat meromorphic connection* on X with poles along Z consists of:

- a vector bundle $\mathcal{E}$ on U, viewed as a locally free module over $\mathcal{O}_X(*Z)$;
- an additive bundle map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}$;

such that:

- the Leibniz rule holds: $\nabla(f\mathbf{v}) = f\nabla(\mathbf{v}) + \mathbf{v} \otimes df$;
- the connection is flat: the induced map $\nabla \circ \nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_{X/K}$ vanishes.

This provides an action on $\mathcal{E}$ of differential operators on X (i.e., a $\mathcal{D}$-module structure).

The goal: formal classification

We wish to classify the possible restrictions of $\mathcal{E}$ to formal local rings, i.e., for $z \in X$, we wish to classify $\mathcal{E} \otimes_{\mathcal{O}_X} \widehat{\mathcal{O}}_{X,z}(*Z)$. This is trivial for $z \in U$, so we may as well take $z \in Z$.

Unfortunately, this is too hard. What's easier is to first replace X by a blowup centered in Z depending on $\mathcal{E}$. The challenge is to exhibit a suitably good blowup *without* starting with a choice of an ideal sheaf to blow up in.

Motivation: even though these formal descriptions are not algebraic or even analytic, they can be used to describe asymptotic behavior of solutions of meromorphic differential equations (Malgrange, Sabbah).

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In the one-dimensional case, we are reduced to considering connections over $K((z))$. These have something like a Jordan decomposition!

Theorem (Turrittin-Levelt-Hukuhara)

Let $\mathcal{E}$ be a connection over $K((z))$. We can then choose a finite extension K' of K and a positive integer m such that $\mathcal{E} \otimes_{K((z))} K'((z^{1/m}))$ decomposes as a direct sum $\bigoplus_i E(s_i) \otimes \mathcal{R}_i$ in which:

- *$E(s_i)$ is free on a generator $\mathbf{v}$ with $\nabla(\mathbf{v}) = \mathbf{v} \otimes ds_i$ for some $s_i \in K'((z^{1/m}))$;*
- *$\mathcal{R}_i$ is regular, i.e., ∇ takes some $K'[[z^{1/m}]]$-lattice $\mathcal{R}_{i,0}$ into $\mathcal{R}_{i,0} \otimes dz/z$.*

Irregularity

In the previous theorem, the quantity

$$\mathrm{irreg}(\mathcal{E}) = \sum_i \mathrm{rank}(\mathcal{R}_i) \cdot \max\{0, -v_z(s_i)\},$$

in which v_z denotes z-adic valuation, is a nonnegative integer independent of choices, the *irregularity* of $\mathcal{E}$. It is zero iff $\mathcal{E}$ is regular.

Picard-Fuchs (Gauss-Manin) connections are always regular (Griffiths). However, interesting irregular connections appear in geometric Langlands (e.g., an example of Gross).

If $\mathcal{E}$ comes from a meromorphic connection on a Riemann surface, the formal decomposition at a point does not descend to the ring of germs of meromorphic functions. But it can be used to form a sector decomposition governing asymptotics of horizontal sections (the *Stokes decomposition*).

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Irregularity and exceptional divisors

Let's return to the original notation, with X of any dimension. For each prime divisor P on X, I can define the *irregularity of $\mathcal{E}$ along P*, either by:

- restricting to a generic curve intersecting P transversely; or
- passing to the generic point η_P of P and choosing a power series representation of the completion of $\operatorname{Frac} \mathcal{O}_{X,\eta}$.

In fact, one also has irregularity along every prime divisor on every blowup of X, (i.e., along each *divisorial valuation* on the function field F of X). (That is, irregularity defines a *b-divisor* on X.)

Coherence of irregularity

Theorem (K, 2011)

There exist a blowup X' of X and a Cartier divisor D on X' supported in the inverse image of Z (the **irregularity divisor***) whose multiplicity along any exceptional divisor on any blowup of X' equals the irregularity of $\mathcal{E}$ along that divisor.*

Theorem (K, 2010)

Choose X' so that both $\mathcal{E}$ and $\mathcal{E}^\vee \otimes \mathcal{E}$ have irregularity divisors on X'. Then over the formal completion at any point of X', $\mathcal{E}$ admits an analogue of the Turrittin-Levelt-Hukuhara decomposition (a **good decomposition** *in the sense of Malgrange).*

Similar results were obtained by totally different methods by Takuro Mochizuki.

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Krull valuations

Let F be the function field of X. A *Krull valuation* on F is a function $v : F \to \Gamma \cup \{+\infty\}$ for some totally ordered group Γ, such that:

(a) For $x \in F$, $v(x) = +\infty$ iff $x = 0$.

(b) For $x, y \in F$, $v(x + y) \geq \min\{v(x), v(y)\}$.

(c) For $x, y \in F$, $v(xy) = v(x) + v(y)$.

We'll also assume that v is trivial on K, i.e., $v(x) = 0$ for $x \in K$.

These include *divisorial valuations*, which measure order of vanishing along exceptional divisors on blowups. But general Krull valuations are much wilder; Γ can be a large subgroup of $\mathbb{R}$, or can even fail to be archimedean!

Riemann-Zariski spaces

The *Riemann-Zariski space* $\mathrm{RZ}(X)$ associated to X is the inverse limit over all blowups $Y \to X$, where each Y is viewed as a *scheme* (i.e., keep all generic points).

This inverse limit is compact for the inverse limit of the constructible topologies (the *patch topology*), and hence quasicompact for the inverse limit of the Zariski topologies (the *Zariski topology*).

We may identify $\mathrm{RZ}(X)$ naturally with a certain set of equivalence class of Krull valuations on F (those satisfying a certain nonnegativity condition; this drops out if X is proper over K).

Working locally

The first theorem I stated can be proved locally on $\mathrm{RZ}(X)$ (using the second theorem). This is easier said than done: for $v \in \mathrm{RZ}(X)$, one can only describe the neighborhoods of v nicely when v is of a special form (an *Abhyankar valuation*).

To get around this, one uses a "fibration in curves" argument in valuation theory. The "curves" here are actually one-dimensional Berkovich nonarchimedean analytic spaces (roughly one-dimensional *tropical spaces*). These show up a lot in number theory...

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Canonicality

The irregularity divisor is only a well-defined element of the *direct limit* of the spaces of Cartier divisors on blowups. Is there a *canonical* construction of a good blowup? (I don't even know how to construct a suitable blowup centered within Z!)

This is needed to transfer results to complex analytic spaces. (Technicals tool used: Grothendieck's *excellent schemes*, Temkin's *functorial resolution of singularities*.)

Links to arithmetic geometry

There are partly understood links between the theory of flat meromorphic connections and arithmetic geometry (especially the study of *p-adic differential equations*, and *wild ramification* of finite covers in positive characteristic).

This setting is a good test case for developing understanding of arithmetic-geometric analogues. For instance, one can use irregularity divisors to compute characteristic cycles of $\mathcal{D}$-modules, and then analogize to positive characteristic (Liang Xiao).

On the other hand, the techniques used here were actually first developed in the arithmetic geometry setting!